Progress in integrable statistical models inspired by supersymmetric gauge theories

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Supersymmetric theories, dualities and deformations

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Based mainly on works with

- Vyacheslav Spiridonov [1505.00765]
- Hjalmar Rosengren [1104.4470]

and works in progress with

Andrew Kels

Also inspired by works of others

[Spiridonov 1011.3798], [Bazhanov and Sergeev 1006.0651; 1106.5874]
[Yamazaki 1307.1128], [Benini, Nishioka, Yamazaki 1109.0283]
[Terashima, Yamazaki 1203.5792], [Yamazaki 1203.5784], [Yamazaki, Yan 1504.05540]
[Kels 1504.07074], [Kels 1302.3025],[Bazhanov, Kels, Sergeev 1602.07076]

Simple Motivation

Integrability is a beautiful phenomenon which plays a very important role in theoretical physics. One of the key structural elements leading to integrability is the Yang-Baxter equation

$$\mathbb{R}_{12}(u-v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{23}(v)=\mathbb{R}_{23}(v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{12}(u-v)$$

The operators $\mathbb{R}_{ik}(u)$ are a function of $u\in\mathbb{C}$ acting in the tensor product $\mathbb{V}\otimes\mathbb{V}\otimes\mathbb{V}$ of some complex vector space $\mathbb{V}$.

The Yang-Baxter equation implies the commutativity of trancfer matrices constructed out of R operator.

Solutions to the Yang-Baxter equation

Recently, there has been observed several connections of integrable statistical models to supersymmetric gauge theories.

One of such connections is a correspondence between supersymmetric quiver gauge theories and integrable lattice models such that the integrability emerges as a manifestation of supersymmetric dualities.

[Spiridonov 1011.3798]

[Benini, Nishioka, Yamazaki 1109.0283]

[Terashima, Yamazaki 1203.5792], [Yamazaki 1203.5784]

Particularly, partition functions of supersymmetric quiver gauge theories can be identified with partition functions of two-dimensional exactly solvable statistical mechanics models in the context of this correspondence.

The idea of the correspondence allows one to obtain

solutions to the Yang-Baxter equation

from supersymmetric gauge theory calculations.

Recent progress

New integrable models from

- $4d$ $\mathcal{N}=1$ duality
  – $S^3 \times S^1$ [Spiridonov 1011.3798]
  – $S^3/Z_r \times S^1$ [Yamazaki 1307.1128], [Kels 1504.07074]
- $3d$ $\mathcal{N}=2$ duality
  – S_b^3 [Spiridonov 1011.3798]
  – S_b^3/Z_r [IG, Kels]
  – $S^2 \times S^1$ [Yagi 1504.04055], [Kels 1504.07074], [IG, Spiridonov 1505.00765]
- $2d$ $\mathcal{N}=(2,2)$ duality
  – $S^1 \times S^1$ [Yamazaki, Yan 1504.05540]
  – S^2 [Kels 1302.3025]

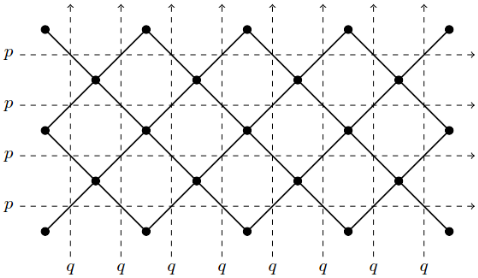
In this talk

I will present new solutions of the Yang-Baxter equation in terms of the **basic and hyperbolic hypergeometric integrals**.

The investigation is restricted to two-dimensional spin models from statistical physics side and to three-dimensional supersymmetric gauge theories from other side of the correspondence.

The new solutions correspond to the **partition functions** of the three-dimensional $\mathcal{N} = 2$ SQCD with six flavors.

Integrable models on the square lattice



- Spin variables

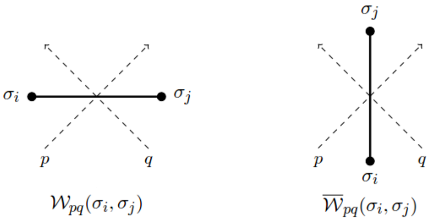
$$\sigma_j = (x_j, m_j)\,, \quad x_j \in \mathbb{R}\,,\; m_j \in \mathbb{Z}\,, \quad j = 1, 2, \ldots, N,$$

are assigned to each vertex of the square lattice.

- Two spins σ_i, σ_j, interact only if they are at vertices i, j, connected by an edge (ij), of the square lattice.

Boltzmann weights

- Interactions are characterized by the Boltzmann weights



- Two edge Boltzmann weights only depend on the difference of rapidity variables $p - q := \alpha$

$$W_\alpha(\sigma_i, \sigma_j) := W_{pq}(\sigma_i, \sigma_j)$$
$$\bar{W}_\alpha(\sigma_i, \sigma_j) := \bar{W}_{pq}(\sigma_i, \sigma_j)$$

Boltzmann weights

- Two Boltzmann weights are also related by the **crossing symmetry**

$$\bar{W}_{\alpha}(\sigma_i, \sigma_j) = W_{\eta-\alpha}(\sigma_i, \sigma_j)$$

  where $\eta > 0$ is a real valued, model dependent **crossing parameter**.
- Boltzmann weights are also spin **reflection symmetric**, such that

$$W_{\alpha}(\sigma_i, \sigma_j) = W_{\alpha}(\sigma_j, \sigma_i)$$

- All lattice models considered here can be interpreted as being **physical**, such that all two-spin interactions are characterised by positive, real valued Boltzmann weights $W_{\alpha}(\sigma_i, \sigma_j)$, that represent real valued interaction energies.

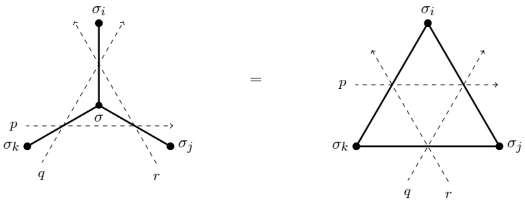
The *partition function* is defined as

$$\mathcal{Z} = \sum \int \prod_{(ij)} W_{\alpha}(\sigma_i, \sigma_j) \prod_{(kl)} W_{\eta-\alpha}(\sigma_k, \sigma_l) \prod_{n} S(\sigma_n)\, dx_n\,.$$

The integral and sum run over all possible values of internal spins in the lattice, and boundary spins are assigned fixed values.

The goal of statistical mechanics is to evaluate the partition function in the thermodynamic limit, as $N \to \infty$.

An exact evaluation is possible if the Boltzmann weights satisfy the Yang-Baxter equation, which for models we considered here takes the form of the following *star-triangle relation*



$$\sum_m \int dx\, S(\sigma) W_{\eta-\alpha_i}(\sigma_i, \sigma) W_{\eta-\alpha_j}(\sigma_j, \sigma) W_{\eta-\alpha_k}(\sigma, \sigma_k)$$
$$= \mathcal{R}(\alpha_i, \alpha_j, \alpha_k)\, W_{\alpha_i}(\sigma_j, \sigma_k) W_{\alpha_j}(\sigma_i, \sigma_k) W_{\alpha_k}(\sigma_j, \sigma_i)$$

New two-dimensional solvable lattice models

Example 1.

- Boltzmann weights

$$\mathrm{W}_a(x, j; u, m) = \Gamma_q(e^{2\pi i(a-\xi\pm x\pm u)}), \quad e^{-4\pi \mathrm{i}\xi} := q$$

- Self interaction

$$S_m(u) = \frac{(1-q^m e^{4\pi i u})(1-q^m e^{-4\pi i u})}{2q^m}$$

- R factor

$$R(a,b) = \frac{(qe^{4\pi i a}, qe^{4\pi i b}, e^{-4\pi i(a+b)}; q)_\infty}{(e^{-4\pi i a}, e^{-4\pi i b}, qe^{4\pi i(a+b)}; q)_\infty}$$

$$\Gamma_q(a, n; z, m) := \frac{(q^{1+\frac{n+m}{2}}\frac{1}{az}, q^{1+\frac{n-m}{2}}\frac{z}{a}; q)_\infty}{a^n z^m (q^{\frac{n+m}{2}} az, q^{\frac{n-m}{2}}\frac{a}{z}; q)_\infty}$$

New two-dimensional solvable lattice models

Example 2.

- Spins

$$\sigma_j = (x_j, m_j)\,, \quad 0 \le x_j < \infty\,, \quad m_j = 0, 1, \ldots, r-1$$

- Boltzmann weights

$$W_\alpha(\sigma_i, \sigma_j) = \frac{1}{\kappa^h(\alpha)} \frac{\varphi_{m_i+m_j}(x_i + x_j + i\alpha)\,\varphi_{m_i-m_j}(x_i - x_j + i\alpha)}{\varphi_{m_i+m_j}(x_i + x_j - i\alpha)\,\varphi_{m_i-m_j}(x_i - x_j - i\alpha)}$$

- Self interaction

$$S(\sigma_j)\frac{1}{r\sqrt{\omega_1\omega_2}}\,\varphi_{-2m_j}(-2x_j - i\eta)\,\varphi_{2m_j}(2x_j - i\eta)$$

$$\varphi_{r,m}(z) = \exp\left(\int_0^\infty \frac{dx}{x}\left(\frac{iz}{\omega_1\omega_2 rx} - \frac{\sinh(2izx - \omega_1(r-2[m]x)}{2\sinh(\omega_1 rx)\sinh(2\eta x)} - \frac{\sinh(2izx + \omega_2(r-2[m]x)}{2\sinh(\omega_2 rx)\sinh(2\eta x)}\right)\right)$$

Interpretation as $3d$ $\mathcal{N} = 2$ theory

- spin lattice — quiver theory with $SU(2)$ gauge groups
- Boltzmann weights — contributions of chiral multiplets to PF
- Self interaction — contribution of vector multiplets to PF
- Star-triangle relation — Seiberg duality

Partition functions on $S^2 \times S^1$ and S^3/Z_r

$3d\ \mathcal{N} = 2$ index

The superconformal index of three-dimensional $\mathcal{N} = 2$ superconformal field theory is a twisted partition function defined on $S^2 \times S^1$ as follows

$$I(q, \{t_i\}) = \mathrm{Tr}\left[(-1)^{\mathrm{F}} e^{-\beta\{Q,Q^\dagger\}} q^{\frac{1}{2}(\Delta+j_3)} \prod_i t_i^{F_i}\right]$$

- the trace is taken over the Hilbert space of the theory on S^2,
- F plays the role of the fermion number which takes value zero on bosons and one on fermions. In presence of monopoles one needs to refine this number by shifting it by $e \times m$, where e and m are electric charge and magnetic monopole charge, respectively.
- Δ is the energy (or conformal dimension via radial quantization), j_3 is the third component of the angular momentum on S^2.
- F_i is the charge of global symmetry with fugacity t_i,

$3d\ \mathcal{N}=2$ index

Q is a certain supersymmetric charge in three-dimensional $\mathcal{N}=2$ superconformal algebra with quantum numbers

$$\Delta = \frac{1}{2}, \ \ j_3 = -\frac{1}{2} \ \text{ and } \ R = 1$$

The supercharges $Q^\dagger = S$ and Q satisfy the anti-commutation relation

$$\frac{1}{2}\{Q, S\} = \Delta - R - j_3$$

- only BPS states with $\Delta - R - j_3 = 0$ contribute to the superconformal index
- the index is β-independent but depends non-trivially on the fugacities t_i and q
- the index counts the number of BPS states weighted by their quantum numbers.

Using the localization technique the superconformal index can be computed exactly it reduces to the following matrix integral

$$I(\underline{t}, \underline{n}; x) = \sum_{m_i} \frac{1}{|W_m|} \int \prod_{i=1}^{\mathrm{rank}G} \frac{dz_i}{2\pi i z_i} Z_{gauge}(z_i, m_i; q) \times \prod_{\Phi} Z_{\Phi}(z_i, m_i; t_a, n_a; q)$$

The squashed lens space $S_b^3/\mathbb{Z}_r$ is defined as the squashed 3-sphere

$$S_b^3 = \{(x,y) \in \mathbb{C}^2 | \ \ b^2|x|^2 + b^{-2}|y|^2 = 1\}$$

with the identification

$$(x,y) \sim (e^{\frac{2\pi i}{r}}x, e^{-\frac{2\pi i}{r}}y)$$

The partition function is decomposed in the following form

$$Z = \sum_m \int \frac{[dz]}{2\pi i \prod_k |\mathcal{W}_k|} \ Z_{\rm cl}[z,m] \ Z_{\rm vector}[z,m] \ Z_{\rm matter}[z,m]$$

The sum is over the holonomies

$$m \ = \ \frac{r}{2\pi} \int_C A_\mu dx^\mu$$

the integration over a non-trivial cycle $\mathbb{C}$ on $S^3/\mathbb{Z}_r$ and A_μ is the gauge field.

Supersymmetric duality

Different theories describe the same physics in their IR fixed points:

- **theory A**: with $SU(2)$ gauge group and quark superfields in the fundamental representation of the $SU(6)$ flavor group.
- **theory B**: does not have gauge degrees of freedom, the matter sector contains meson superfields in 15-dimensional antisymmetric $SU(6)$-tensor representation of the second rank

[Seiberg 9402044]

The duality between theories A and B leads to the equality of corresponding partition functions.

Basic hypergeometric integral identity

Superconfromal indices:

$$\sum_{m\in\mathbb{Z}}\int_{\mathbb{T}} q^{-|m|}\prod_{j=1}^{6}\frac{(q^{1+\frac{n_j}{2}+\frac{|m|}{2}}\frac{1}{a_j z}, q^{1+\frac{n_j}{2}+\frac{|m|}{2}}\frac{z}{a_j}; q)_\infty}{(q^{\frac{n_j}{2}+\frac{|m|}{2}}a_j z, q^{\frac{n_j}{2}+\frac{|m|}{2}}\frac{a_j}{z}; q)_\infty}(1-q^{|m|}z^2)(1-q^{|m|}z^{-2})\frac{dz}{2\pi i z}$$
$$=\frac{1}{\prod_{j=1}^{6}a_j^{n_j}}\prod_{1\le j<k\le 6}\frac{(q^{1+\frac{n_j}{2}+\frac{n_k}{2}}a_j^{-1}a_k^{-1}; q)_\infty}{(q^{\frac{n_j}{2}+\frac{n_k}{2}}a_j a_k; q)_\infty}$$

with the balancing conditions

$$\prod_{i=1}^{6} a_i = q, \quad \text{and} \quad \sum_{i=1}^{6} n_i = 0$$

mathematical proof [IG, Rosengren 1606.08185]

[IG, Spiridonov 1505.00765]

Hyperbolic hypergeometric integral identity

Lens indices

$$\sum_{m_0=0}^{r-1}\int_{\mathbb{R}}\frac{dx_0}{2\pi i}\,4\sinh\frac{2\pi}{r\omega_1}(x_0-i\omega_1 m_0)\sinh\frac{2\pi}{r\omega_2}(x_0+i\omega_2 m_0)$$

$$\times\prod_{k=1}^{6}\frac{\hat{s}_{b,-m_0-m_k}(x_0-x_k+iQ/2)}{\hat{s}_{b,-m_0+\bar{m}_k}(x_0+x_k-iQ/2)}\;=\prod_{1\le j<k\le 6}\hat{s}_{b,-m_j-m_k}(x_j+x_k+iQ)$$

with the balancing condition

$$\sum_{i=1}^{6}x_i=Q$$

[IG, Kels]

By adding a certain superpotential one may break flavor symmetry of both theories from $SU(6)$ down to $SU(2) \times SU(2) \times SU(2)$. Then the integral identities get the form of the star-triangle relation.

Example 1.

$$\sum_{m\in\mathbb{Z}} \int [d_m z] \Gamma_q(q^{\frac{1}{6}} t/sa^{\pm 1}, \pm A; z, m) \Gamma_q(q^{\frac{1}{6}} s/rb^{\pm 1}, \pm B; z, m) \Gamma_q(q^{\frac{1}{6}} r/tc^{\pm 1}, \pm C; z, m)$$

$$= \frac{(q^{\frac{2}{3}}(t/s)^{-2}, q^{\frac{2}{3}}(s/r)^{-2}, q^{\frac{2}{3}}(r/t)^{-2}; q)_\infty}{(q^{\frac{1}{3}}(t/s)^2, q^{\frac{1}{3}}(s/r)^2, q^{\frac{1}{3}}(r/t)^2; q)_\infty} \Gamma_q(q^{\frac{1}{3}} t/ra^{\pm 1}, \pm A; b, B)$$

$$\times \Gamma_q(q^{\frac{1}{3}} r/sc^{\pm 1}, \pm C; a, A) \Gamma_q(q^{\frac{1}{3}} s/tb^{\pm 1}, \pm B; c, C)$$

R-**matrix**

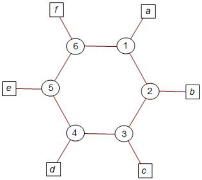
$$R_{(m,l)(n,r)} \left(\begin{array}{cc} a, A & b, B \\ d, D & c, C \end{array} \right) = \frac{(q^{\frac{2}{3}}(n/l)^{-2}, q^{\frac{2}{3}}(r/m)^{-2}; q)_\infty}{(q^{\frac{1}{3}}(n/l)^2, q^{\frac{1}{3}}(r/m)^2; q)_\infty} \sum_{k\in\mathbb{Z}} \int [d_k z]$$

$$\times \Gamma_q(q^{\frac{1}{3}} \frac{l}{n} a^{\pm 1}, \pm A; z, k) \Gamma_q(q^{\frac{1}{6}} \frac{r}{l} b^{\pm 1}, \pm B; z, k)$$

$$\times \Gamma_q(q^{\frac{1}{3}} \frac{m}{r} c^{\pm 1}, \pm C; z, k) \Gamma_q(q^{\frac{1}{6}} \frac{n}{m} d^{\pm 1}, \pm D; z, k)$$

IRF type Yang-Baxter

Using star-triangle relation one can obtain IRF Yang-Baxter from the following quiver diagram



Summary

- We presented new solutions to the star-triangle relation (Yang-Baxter equation) expressed in terms of basic and hyperbolic hypergeometric functions. The new solutions correspond to new solvable two-dimensional lattice models of statistical mechanics.
- One can obtain the known models when a temperature-like parameter q tends to one in our solutions.
- The R matrix is dictated by some quantum group. We wish to elucidate the origin of our solutions in the framework of the representation theory of quantum group.
- Other Seiberg dualities
- There are a lot of attempts to extend the idea of integrability to three-dimensional lattice models. The Yang-Baxter equation in this case takes the form of the so-called tetrahedron equation by Zamolodchikov. It would be interesting to extend the relationship between supersymmetric dualities and integrable models and find a solution of the tetrahedron equation in this context.