Coulomb branches for rank two gauge groups in 3d $\mathcal{N} = 4$ gauge theories

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July 6, 2016

Based on collaboration [arXiv:1605.00010] with A. Hanany

Why Coulomb branches of 3d $\mathcal{N} = 4$ gauge theories?

Physics point of view

- Coulomb branch affected by **quantum corrections**
  - abelian theories understood
  - "some" quiver gauge theories understood via branes
  - Q: what about generic non-abelian theory?
- study Coulomb branch from **algebraic perspective**

  $\longrightarrow$ Hilbert series

Why Coulomb branches of 3d $\mathcal{N} = 4$ gauge theories?

Maths point of view

- **hyper-Kähler geometry**
  - hyper-Kähler quotient $\rightarrow$ e.g. Higgs branch
  - **“other means”** $\rightarrow$ e.g. Coulomb branch

Outline

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3d $\mathcal{N} = 4$ gauge theories

Defined by

- choice of **gauge group**: a compact Lie group G
- choice of **matter content**: a representation $\mathcal{R}$

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Bosonic field content

- **vector multiplet**: (v-plet)

$$\left\{ \begin{matrix} \mathcal{N}{=}4 \text{ v-plet} \\ (A, \phi_1, \phi_2, \phi_3) \end{matrix} \right\} = \left\{ \begin{matrix} \mathcal{N}{=}2 \text{ v-plet} \\ (A, \sigma \equiv \phi_3) \end{matrix} \right\} + \left\{ \begin{matrix} \mathcal{N}{=}2 \text{ chiral} \\ (\Phi \equiv \phi_1 + \mathrm{i}\phi_2) \end{matrix} \right\}$$

- **hypermultiplet**: (h-plet)

  $4N$ real scalars for some $N \geqslant 0$

Coulomb branch

Monopole operators [Borokhov, Kapustin & Wu]

$\longrightarrow$ generate entire chiral ring

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- **Dirac monopole singularity** for A

$$A_{\pm} \propto \frac{m}{2} \left(\pm 1 - \cos\theta \right) \mathrm{d}\varphi$$

with *Dirac quantisation condition* $\exp\left(2\pi \mathrm{i} m\right) = \mathbb{1}_{\mathrm{G}}$

$\rightarrow$ GNO-dual group $\widehat{\mathrm{G}}$

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- by $\mathcal{N} = 4$ supersymmetry: $\Phi \in \mathrm{Lie}(\mathrm{H}_m)$
  - if $\Phi = 0$ then bare monopole operator
  - if $\Phi \neq 0$ then dressed monopole operator

Monopole formula

Objective: count all bare and dressed monopole operators

→ proposed by [Cremonesi, Hanany & Zaffaroni]

$$\mathrm{HS}_{\mathrm{G}}(t,z) = \sum_{m \in \Lambda_w(\hat{\mathrm{G}})/\mathcal{W}_{\hat{\mathrm{G}}}} z^{J(m)} t^{\boldsymbol{\Delta}(m)} P_{\mathrm{G}}(t,m)$$

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- $\mathbf{SU(2)}_{\boldsymbol{R}}$ **isospin**: conformal dimension

$$\Delta(m) = \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho\in\mathcal{R}_i}|\rho(m)| - \sum_{\alpha\in\Phi_+}|\alpha(m)|$$

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- global symmetries charges: $J(m)$

Outline

Cones, fans, and monoids —1—

Aim: $\mathrm{HS}_{\mathrm{G}}(t,z) = \sum_{m\in\Lambda_w(\hat{\mathrm{G}})/\mathcal{W}_{\hat{\mathrm{G}}}} z^{J(m)} t^{\Delta(m)} P_{\mathrm{G}}(t,m)$

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- $\Lambda_w(\widehat{\mathrm{G}})/\mathcal{W}_{\widehat{\mathrm{G}}}$ is a **monoid** of dominant Weyl chamber $\sigma_{\widehat{\mathrm{G}}}$

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- positive grading Δ for **good theories** [Gaiotto & Witten]

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- positive grading Δ for good theories [Gaiotto & Witten]
- summand: t^{Δ} contains $|\rho(m)|$

  $\longrightarrow$ split summation range

Cones, fans, and monoids —2—

- For each weight μ in Δ define

hyperplane $H_\mu := \{m \in \mathfrak{t} \mid \mu(m) = 0\} \subseteq \mathfrak{t}$,

half-space $H_\mu^\pm := \{m \in \mathfrak{t} \mid \mu(m) \gtrless 0\} \subseteq \mathfrak{t}$.

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- All absolute values resolved on each

$$\sigma_{\epsilon_1,\epsilon_2,\ldots,\epsilon_Q} := H_{\mu_1}^{\epsilon_1} \cap H_{\mu_2}^{\epsilon_2} \cap \cdots \cap H_{\mu_Q}^{\epsilon_Q} \subset \mathfrak{t}$$
$$\text{with} \quad \epsilon_i = \pm \qquad \text{for} \quad i = 1, \ldots, Q\,.$$

$\rightarrow$ **polyhedral cones**

Γ={ all weight in Δ which are not multiples },

Q=$|\Gamma|$, and $\mu_i \in \Gamma$

Cones, fans, and monoids —3—

- **Restrict** to dominant Weyl chamber $\sigma_{\widehat{\mathrm{G}}}$

$$C_p := \sigma_{\epsilon_1,\epsilon_2,\ldots,\epsilon_Q} \cap \sigma_{\widehat{\mathrm{G}}} \qquad \text{with} \quad p = (\epsilon_1, \epsilon_2, \ldots, \epsilon_Q)\ .$$

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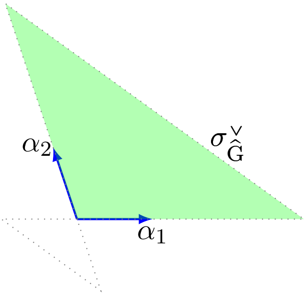

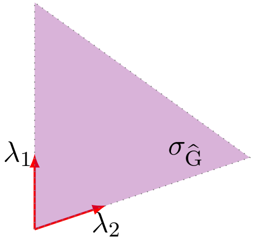

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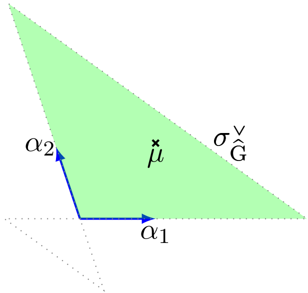

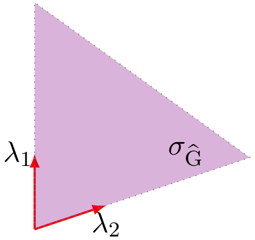

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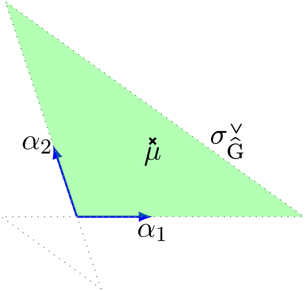

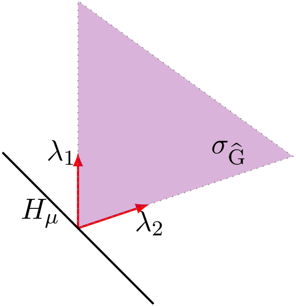

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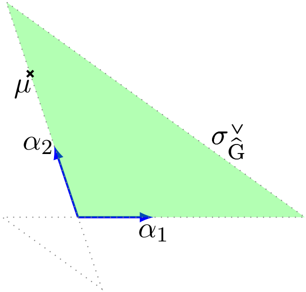

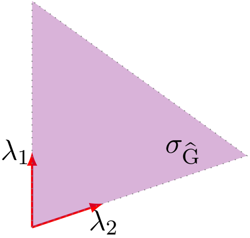

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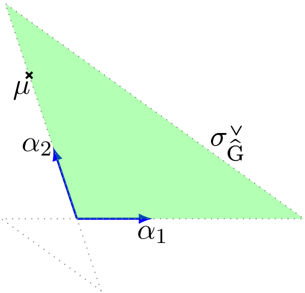

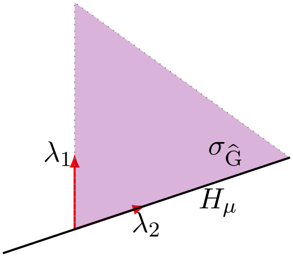

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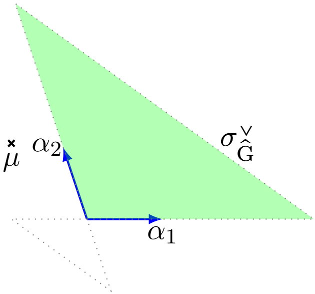

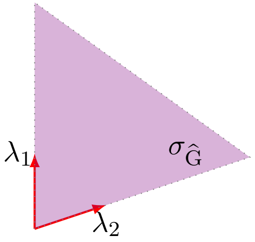

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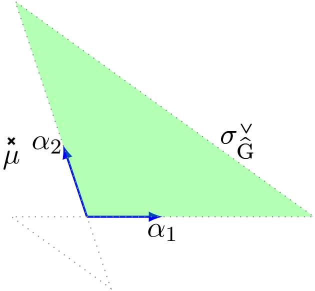

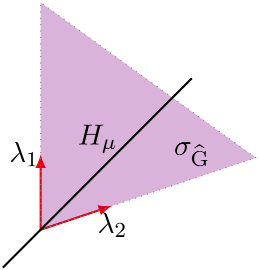

Cones, fans, and monoids —4—

Relevant weights

$\mu \in \Gamma$ leads to hyperplane intersecting Weyl chamber of $\widehat{\mathrm{G}}$ non-trivially $\Leftrightarrow$ neither μ nor $-\mu$ lies in rational cone spanned by simple roots $\mathbf{\Phi}_s$ of G.

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- Restrict to magnetic weight lattice $\rightarrow$ **monoids**

$$S_p := C_p \cap \Lambda_w(\widehat{\mathrm{G}}) \qquad \text{for} \quad p \in I$$

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  - **Gordan's lemma**: S_p are finitely generated
  - Exists unique minimal generating set of S_p, $\Longrightarrow$ the **Hilbert basis** $\mathcal{H}(S_p)$

- Summation range: **monoids** S_p, $p \in I$

Cones, fans, and monoids —5—

- Summation range: **monoids** S_p, $p \in I$
- Cones constitute a **fan** $\boldsymbol{F_\Delta = \{C_p \,|\, p \in I\}}$

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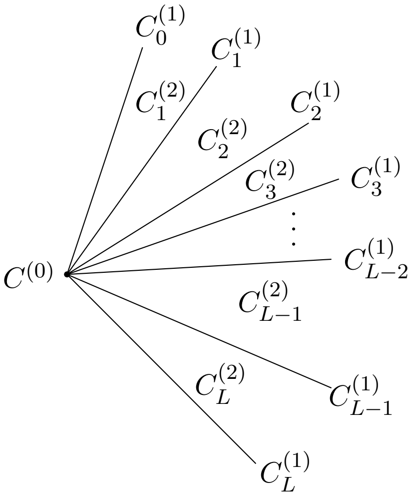
Claim

Collection $\{\mathcal{H}(S_p) \,|\, p \in I\}$ of Hilbert bases is the necessary set of (bare) monopole operators for a theory with conformal dimension Δ.

Summing the Hilbert series —1—

For simplicity: rank two gauge group



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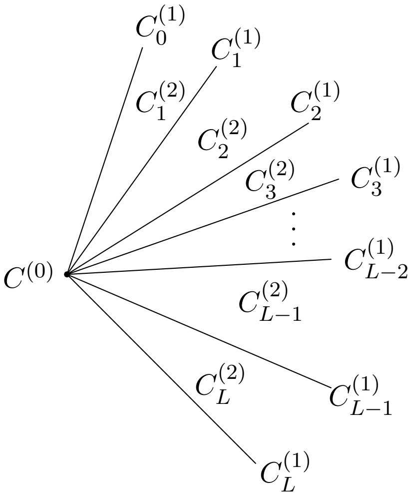

Hilbert bases:

$\mathcal{H}(S_p^{(1)}) = \{x_p\}$

$\rightarrow$ ray generators

$\mathcal{H}(S_p^{(2)}) = \{x_{p-1}, x_p, \{u_\kappa^p\}_\kappa\}$

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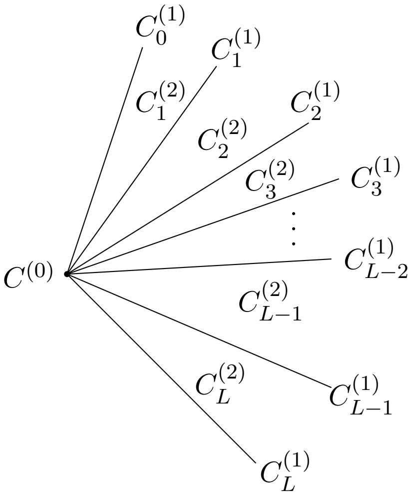

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$s \in S_p^{(2)}$ has representation

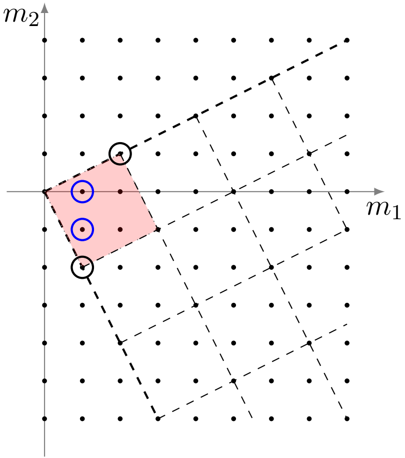
$$s = a_0 x_{p-1} + a_1 x_p + \sum_\kappa b_\kappa u_\kappa^p$$

but not unique

$\rightarrow$ be careful

Summing the Hilbert series —2—

Q: How to sum over monoid?



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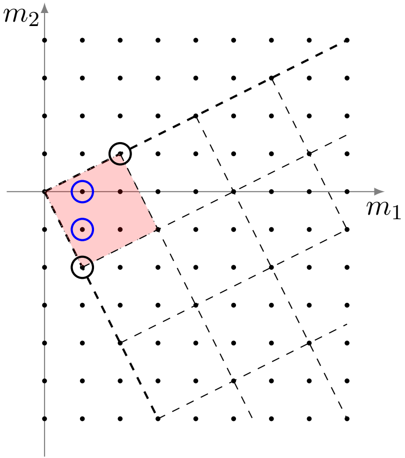

fundamental parallelotop

$$\mathcal{P}(C_p^{(2)}) := \{a_0 x_{p-1} + a_1 x_p \mid 0 < a_0, a_1 < 1\}$$

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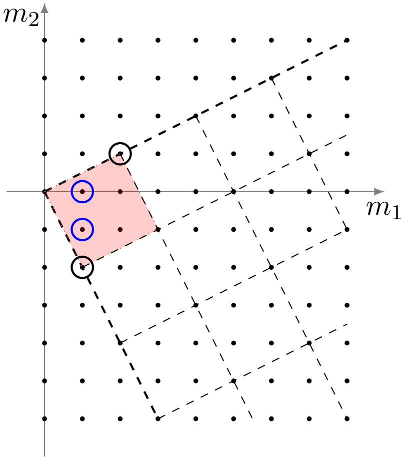

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- sum sub-monoid spanned by x_{p-1} and x_p
- transport $\mathcal{P}(C_p^{(2)})$ along sub-monoid

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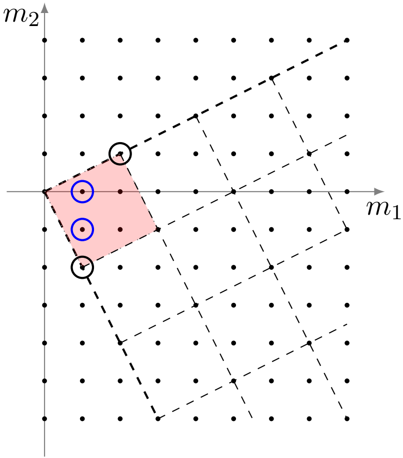

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$\mathcal{P}(C_p^{(2)})$ contains $\#_p := |\det(x_{p-1}, x_p)| - 1$ points

Summing the Hilbert series —3—

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$$+ \sum_{p=1}^{L} \underbrace{\sum_{n_{p-1},n_p>0} P_{\mathrm{G}}(t,x_{p-1}+x_p) t^{\Delta(n_{p-1}x_{p-1}+n_p x_p)}}_{\text{sub-monoid spanned by } x_{p-1} \text{ and } x_p}$$

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$$\mathrm{HS}_{\mathrm{G}}(t) = \sum_{m \in \Lambda_w(\hat{\mathrm{G}})/\mathcal{W}_{\hat{\mathrm{G}}}} t^{\Delta(m)} P_{\mathrm{G}}(t, m)$$

$$= \underbrace{P_{\mathrm{G}}(t, 0)}_{\text{0-dim. monoid}} + \sum_{p=0}^{L} \underbrace{P_{\mathrm{G}}(t, x_p) \sum_{n_p > 0} t^{n_p \Delta(x_p)}}_{\text{1-dim. monoids}}$$

$$+ \sum_{p=1}^{L} \underbrace{\sum_{n_{p-1}, n_p > 0} P_{\mathrm{G}}(t, x_{p-1} + x_p) t^{\Delta(n_{p-1} x_{p-1} + n_p x_p)}}_{\text{sub-monoid spanned by } x_{p-1} \text{ and } x_p}$$

$$+ \sum_{p=1}^{L} \underbrace{\sum_{s \in \#_p} \sum_{n_{p-1}, n_p \geqslant 0} P_{\mathrm{G}}(t, s) t^{\Delta(s + n_{p-1} x_{p-1} + n_p x_p)}}_{\text{transport fundamental parallelotop around}}$$

Summing the Hilbert series —4—

$$\mathrm{HS}_{\mathrm{G}} = \frac{P_{\mathrm{G}}(t,0)}{\prod_{p=0}^{L}\left(1-t^{\Delta(x_p)}\right)} \Bigg\{ \prod_{q=0}^{L}\left(1-t^{\Delta(x_q)}\right) + \sum_{q=0}^{L} \frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} t^{\Delta(x_q)} \prod_{\substack{r=0\\ r\neq q}}^{L}\left(1-t^{\Delta(x_r)}\right) + \sum_{q=1}^{L} \frac{P_{\mathrm{G}}(t,C_q^{(2)})}{P_{\mathrm{G}}(t,0)} \left[t^{\Delta(x_{q-1})+\Delta(x_q)} + \sum_{s\in\#_q} t^{\Delta(s)} \right] \prod_{\substack{r=0\\ r\neq q-1,q}}^{L}\left(1-t^{\Delta(x_r)}\right) \Bigg\}$$

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$$\mathrm{HS}_{\mathrm{G}}=\frac{P_{\mathrm{G}}(t,0)}{\prod_{p=0}^{L}\left(1-t^{\Delta(x_p)}\right)}\Bigg\{\prod_{q=0}^{L}\left(1-t^{\Delta(x_q)}\right)$$

$$+\sum_{q=0}^{L}\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)}t^{\Delta(x_q)}\prod_{\substack{r=0\\ r\neq q}}^{L}\left(1-t^{\Delta(x_r)}\right)$$

$$+\sum_{q=1}^{L}\frac{P_{\mathrm{G}}(t,C_q^{(2)})}{P_{\mathrm{G}}(t,0)}\left[t^{\Delta(x_{q-1})+\Delta(x_q)}+\sum_{s\in\#_q}t^{\Delta(s)}\right]\prod_{\substack{r=0\\ r\neq q-1,q}}^{L}\left(1-t^{\Delta(x_r)}\right)\Bigg\}$$

Rational function with

- denominator $P_{\mathrm{G}}(t,0)/\prod_{p=0}^{L}\left(1-t^{\Delta(x_p)}\right)$
- numerator $\{\ldots\}$

Summing the Hilbert series —5—

Observations:

- Apparent **poles**: Casimirs of G and ray generators x_q

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- Hilbert series determined by **finite set of numbers**
- Same procedure for refined Hilbert series

Outline

1. Set-up
2. Analysis of monopole formula
   - Cones, fans, and monoids
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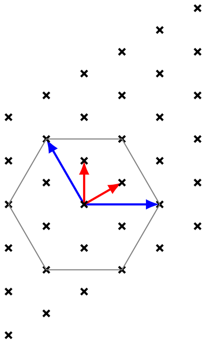
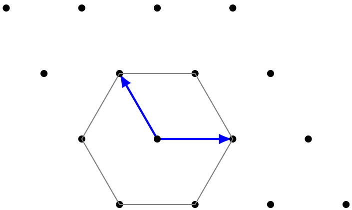
3. Example
   - $\mathrm{SU}(3)$ with N fundamentals
4. Conclusions and outlook

$SU(3)$ gauge theory with N fundamentals

gauge group $SU(3)$



GNO-dual $SU(3)/\mathbb{Z}_3 = PSU(3)$



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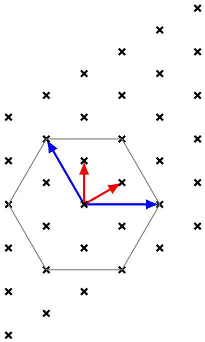

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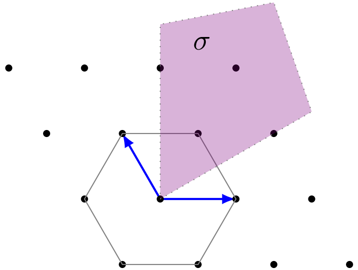

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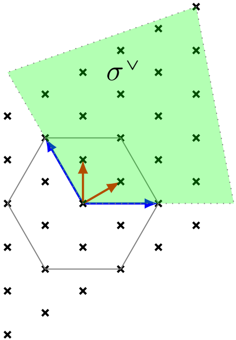

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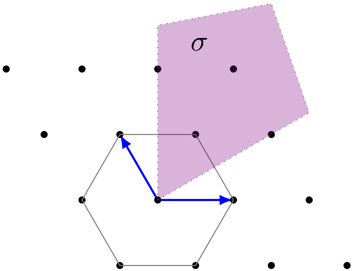

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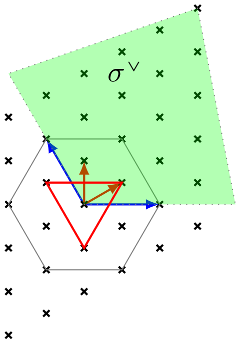

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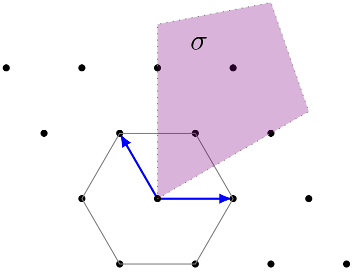

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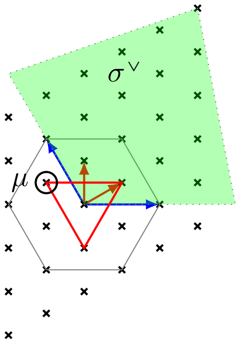

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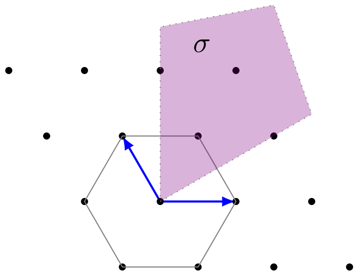

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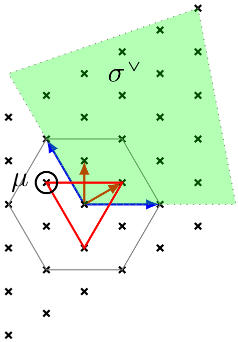

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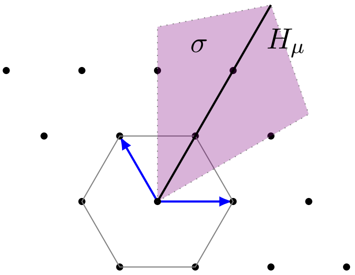

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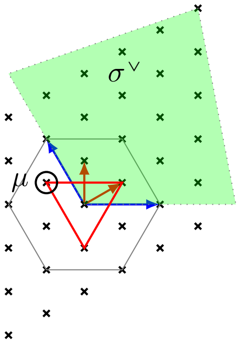

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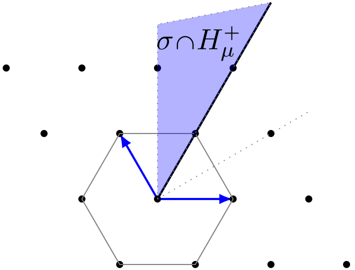

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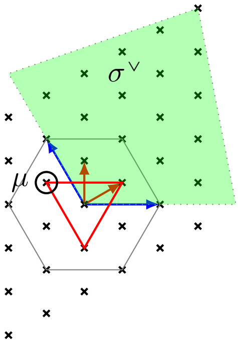

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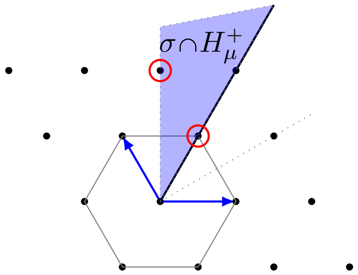

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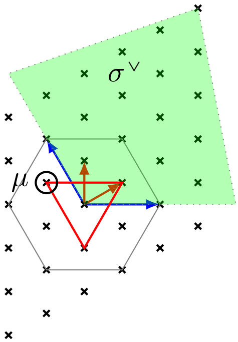

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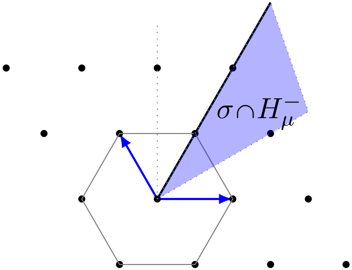

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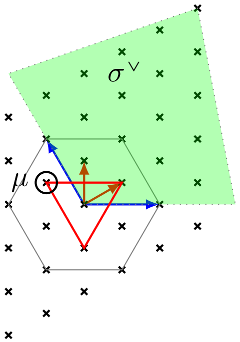

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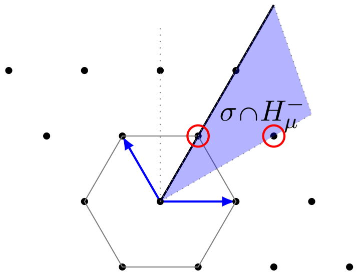

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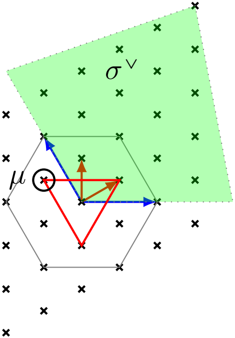

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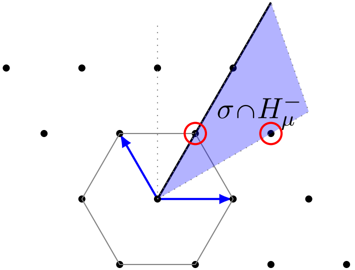

$\Longrightarrow$ expect **three bare monopole generators**

SU(3) gauge theory with N fundamentals

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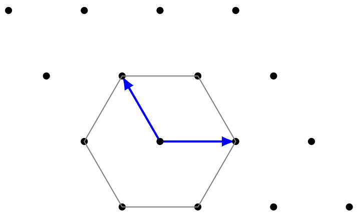

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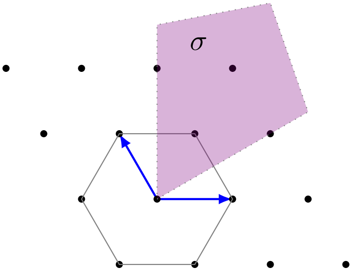

$$\mathrm{H}_m = \mathrm{U}(1) \times \mathrm{U}(1)$$

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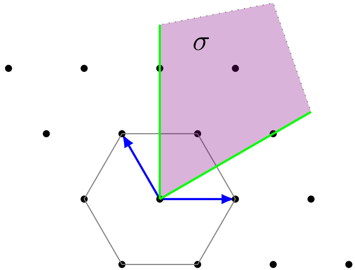

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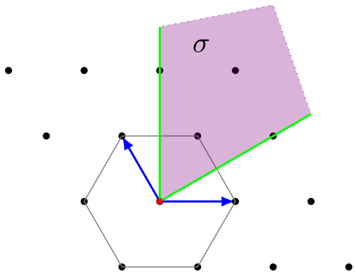

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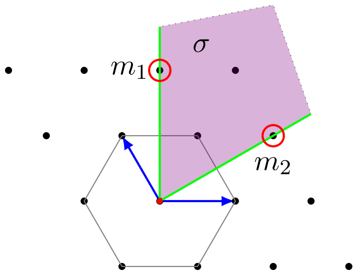

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SU(3) gauge theory with N fundamentals

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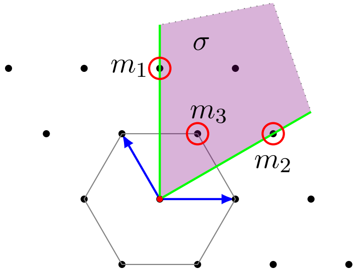

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$$\frac{\#\text{ dressings}}{+1\text{ bare}}\Big|_{m_3} = \frac{|\mathcal{W}_{\mathrm{SU}(3)}|}{|\mathcal{W}_{\mathrm{U}(1)\times\mathrm{U}(1)}|} = 6$$

SU(3) gauge theory with N fundamentals

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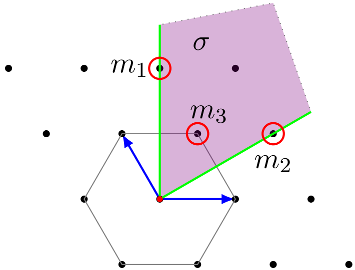

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$\implies$ expect **12 monopole generators**

SU(3) gauge theory with N fundamentals

Hilbert series

$$\mathrm{HS} = \frac{1 + t^{N-3}(2 + 2t + t^2) + t^{2N-6}(1 + 2t + 2t^2) + t^{3N-7}}{(1 - t^2)(1 - t^3)(1 - t^{N-4})(1 - t^{2N-6})}$$

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$\longrightarrow$ Casimir invariants SU(3)

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$\longrightarrow$ bare and dressed monopoles for m_1 and m_2

$\mathrm{SU}(3)$ gauge theory with N fundamentals

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$\longrightarrow$ bare and dressed monopoles for m_3

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Conclusions

Understood impact of choices . . .

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gauge group

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  $\longrightarrow$ monoid

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matter content

- fan within Weyl chamber

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  $\rightarrow$ graded by Δ

$\Longrightarrow$ **intertwine to monopole formula**

Conclusions

Now clear how to . . .

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- rearrange and sum the monopole formula.

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Now clear how to ...

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  $\longleftrightarrow$ fractions of Poincare series $P_{\mathrm{G}}(t,m)$.

Conclusions

Now clear how to ...

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- select **bare monopole generators**

  $\longleftrightarrow$ Hilbert bases for monoids.
- determine number and degree of **dressed monopole operators**

  $\longleftrightarrow$ fractions of Poincare series $P_{\mathrm{G}}(t,m)$.
- proceed with higher rank gauge groups.

Outlook

Geometric picture suggests ...

- Hilbert series of monoids $\longrightarrow$ commutative algebra
- $\mathcal{M}_C$ "patched together" from coordinate rings of monoids?
- might lead to insights on relations between generators.