$T[SU(N)]$ duality webs II: Gauge/CFT correspondence and $2d$ limit

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based on 1712.08140 with Pasquetti and Zenkevich
and work in progress with Aprile, Pasquetti, Sacchi and Zenkevich

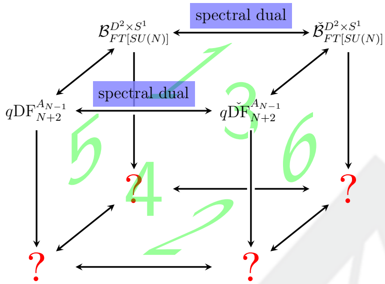
Introduction

$$\begin{tikzcd}
 & Z^{5d}_{U(N)^{N-1}} \arrow[rr, leftrightarrow, "S\text{-dual}"] \arrow[dd] \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow, bend left, "3"] & & \check{Z}^{5d}_{U(N)^{N-1}} \arrow[dd] \arrow[dl, leftrightarrow] \\
\langle V_1 \cdots V_{N+2} \rangle_{qA_{N-1}} \arrow[rr, leftrightarrow, "\text{spectral dual}" , "2"'] \arrow[dd, "5"] \arrow[urrr, leftrightarrow, bend left, "\text{AGT}"] & & \langle \check{V}_1 \cdots \check{V}_{N+2} \rangle_{qA_{N-1}} \arrow[dd, "4"] & \\
 & \mathcal{B}^{D^2 \times S^1}_{FT[SU(N)]} \arrow[rr, leftrightarrow, "\text{spectral} \quad \text{dual}"] \arrow[from=uu, phantom, "6"] & & \check{\mathcal{B}}^{D^2 \times S^1}_{FT[SU(N)]} \\
q\mathrm{DF}^{A_{N-1}}_{N+2} \arrow[rr, leftrightarrow, "\text{spectral dual}", "1"'] \arrow[ur, leftrightarrow] & & q\check{\mathrm{DF}}^{A_{N-1}}_{N+2} \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

Summary of Saras talk:

- $FT[SU(N)]$ theory and its spectral dual $FT[SU(N)]^{\vee}$.
- $FT[SU(N)]$ as a codimension-two defect theory.
- $3d$ spectral duality from fiber-base duality via Higgsing.

Introduction



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 & ? \arrow[rr, leftrightarrow] \arrow[dl, leftrightarrow] & & ? \arrow[dl, leftrightarrow] \\
? \arrow[rr, leftrightarrow] & & ? &
\end{tikzcd}$$

In this talk:

- Forget about $5d$ web.
- Details of $3d$ gauge/CFT correspondence (face 1)
- *What is duality web obtained after circle reduction of $3d$ web?* (face2).

Outline

- Gauge/CFT correspondence
  - Toda CFT
  - Dotsenko-Fateev representation of the conformal blocks
  - *q*-deformation and gauge/CFT correpondence
- 2*d* limit of 3*d* duality web
  - Coulomb and Higgs limits
  - Reduction of mirror dual holomorphic blocks
  - Bonus: *d*-Virasoro and its conformal blocks
- Conclusions and Outlook

Part I:

3*d* gauge/CFT duality web

$$\begin{tikzcd}
\mathcal{B}^{D_2\times S^1}_{FT[SU(N)]} \arrow[r, leftrightarrow, "\text{spectral dual}"] \arrow[d, leftrightarrow, "\text{gauge}/q\text{DF}"'] & \check{\mathcal{B}}^{D_2\times S^1}_{FT[SU(N)]} \arrow[d, leftrightarrow, "\text{gauge}/q\text{DF}"] \\
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A_n Toda CFT

$$S_{\text{Toda}} = \int d^2z\,\sqrt{g}\left[\frac{1}{8\pi}g^{z\bar{z}}(\partial_z\vec{\phi},\partial_{\bar{z}}\vec{\phi}) + \frac{1}{4\pi}\mathcal{Q}_\beta(\vec{\rho},\vec{\phi})\,R + \mu\sum_{a=1}^{N} e^{\sqrt{\beta}(\vec{\phi},\,\vec{e}_{(a)})}\right]$$

A_n Toda CFT

Kinetic term

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- $\vec{\phi} = \left(\phi^{(1)},\ldots,\phi^{(n+1)}\right)$ - $(n+1)$ component scalar field

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$$\mathcal{Q}_\beta = \beta^{1/2} - \beta^{-1/2}$$

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$$c = n - n(n+1)(n+2)\mathcal{Q}_\beta^2$$

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- *Primary vertex operator* $\quad V_{\vec{\alpha}}(z) = \;: e^{\frac{1}{\sqrt{\beta}}(\vec{\alpha},\vec{\phi}(z))}: \quad$ $\vec{\alpha}$ is momentum.

Conformal Blocks of Toda

- We are interested in *correlators of primary vertex operators*

$$\langle V_{\vec{\alpha}^{(\infty)}}(\infty)\, V_{\vec{\alpha}^{(1)}}\,(z_1)\ldots V_{\vec{\alpha}^{(l)}}\,(z_l)\;\; V_{\vec{\alpha}^{(0)}}(0)\rangle_{\mathrm{Toda}}$$

Very hard due to the exponential Toda potential!

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- *Solution: treat potential perturbatively*. This leads to the following expressions for the *conformal blocks* of TFT

$$\mathrm{DF}^{A_n}_{l+2} = \langle\vec{\alpha}^{(\infty)}|\, V_{\vec{\alpha}^{(1)}}(z_1)\dots V_{\vec{\alpha}^{(l)}}(z_l)\prod_{a=1}^{n} Q_{(a)}^{N_a}|\vec{\alpha}^{(0)}\rangle_{\mathrm{free}}$$

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Correlator in theory of (n+1) free bosons.

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$$Q_{(a)} = \oint dx S_{(a)}(x)$$

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External momentum state $P^{(a)}|\vec{\alpha}\rangle = \frac{1}{\sqrt{\beta}}\alpha_a|\vec{\alpha}\rangle$. Equiv. insertion of $V_{\vec{\alpha}}$ at ∞ or 0.

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Correlator in theory of (n+1) free bosons.

- Free field correlators are non-trivial only if *momenta conservation condition* is satisfied

$$\boxed{2\sqrt{\beta}\mathcal{Q}_\beta\vec\rho = \vec\alpha^{(0)} + \vec\alpha^{(\infty)} + \sum_{j=1}^{l}\vec\alpha^{(j)} + \beta\sum_{a=1}^{n}N_a\vec e_{(a)}}$$

Dotsenko, Fateev '84; Fateev,Litvinov '07

DF integrals

- Computing free field correlator ➡ simple *normal ordering*

DF integrals

- Computing free field correlator → simple *normal ordering*

$$\mathrm{DF}^{A_n}_{l+2} \sim$$

$$\oint \prod_{a=1}^{n} \prod_{i=1}^{N_a} dx_i^{(a)} \left(x_i^{(a)}\right)^{\beta(N_a - N_{a+1} - 1) + (\alpha_a^{(0)} - \alpha_{a+1}^{(0)})} \prod_{a=1}^{n} \prod_{i \neq j}^{N_a} \left(1 - \frac{x_j^{(a)}}{x_i^{(a)}}\right)^{\beta}$$

$$\prod_{a=1}^{n-1} \prod_{i=1}^{N_a} \prod_{j=1}^{N_{a+1}} \left(1 - \frac{x_j^{(a+1)}}{x_i^{(a)}}\right)^{-\beta} \times \prod_{p=1}^{l} \prod_{a=1}^{n} \prod_{i=1}^{N_a} \left(1 - \frac{x_i^{(a)}}{z_p}\right)^{\alpha_a^{(p)} - \alpha_{a+1}^{(p)}}$$

DF integrals

- Computing free field correlator ➡ simple *normal ordering*

$$\mathrm{DF}^{A_n}_{l+2} \sim \oint \prod_{a=1}^{n}\prod_{i=1}^{N_a} dx_i^{(a)} \overbrace{\left(x_i^{(a)}\right)^{\beta(N_a-N_{a+1}-1)+(\alpha_a^{(0)}-\alpha_{a+1}^{(0)})}}^{S(x)|\vec{\alpha}^{(0)}\rangle} \overbrace{\prod_{a=1}^{n}\prod_{i\neq j}^{N_a}\left(1-\frac{x_j^{(a)}}{x_i^{(a)}}\right)^{\beta}}^{:S_{(a)}S_{(a)}:}$$

$$\underbrace{\prod_{a=1}^{n-1}\prod_{i=1}^{N_a}\prod_{j=1}^{N_{a+1}}\left(1-\frac{x_j^{(a+1)}}{x_i^{(a)}}\right)^{-\beta}}_{:S_{(a)}S_{(a+1)}:} \times \underbrace{\prod_{p=1}^{l}\prod_{a=1}^{n}\prod_{i=1}^{N_a}\left(1-\frac{x_i^{(a)}}{z_p}\right)^{\alpha_a^{(p)}-\alpha_{a+1}^{(p)}}}_{:S_{(a)}(x)V_{\vec{\alpha}}(z):}$$

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- In Liouville case $(n=1)$ - *Selberg integral*, can be evaluated
- For Toda case $(n>1)$ answers are known only for the cases of 2 arbitrary + (semi-)degenerate operators.

Degenerate: $\boxed{\vec{\alpha}=\beta\vec{\omega}_n}$ *Semi-degenerate:* $\boxed{\vec{\alpha}=\kappa\vec{\omega}_n}$

q-Virasoro/qW_n algebra

Shiraishi, Kubo, Awata, Odake '95; Frenkel, Reshetikhin '95;

Generators T_n of $\mathcal{V}ir_{q,t}$ algebra satisfy commutation relations

$$[T_n, T_m] = -\sum_\ell f_\ell (T_{n-\ell} T_{m+\ell} - T_{m-\ell} T_{n+\ell}) - \frac{(1-q)(1-t^{-1})}{(1-v^2)} (v^{2n} - v^{-2n}) \delta_{n+m,0}$$

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- Deformation is parametrized by $q\,, t = q^\beta \in \mathbb{C}$, $v = \sqrt{q\,t^{-1}}$.
- *Structure constants* f_l fixed by associativity

$$\sum_{\ell>0} f_\ell\, z^\ell = \exp\left(\sum_{n>0} \frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{(1+v^{2n})} z^n\right)$$

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- *Virasoro limit:* $q = e^\hbar\,,\ \hbar \to 0\,.$ β and z are fixed!

$$T_n \to 2\delta_{n,0} + \beta\hbar^2 \left(\hat{L}_n + \frac{1}{4}\mathcal{Q}_\beta^2\, \delta_{n,0}\right) + O\left(\hbar^4\right)$$

$\hat{L}_n$ - Virasoro generator with a central charge $c = 1 - 6\mathcal{Q}_\beta^2$

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- Generalization to $q\mathcal{W}_n$ case is derived using *bosonization*.

Bosonization of qW_n algebra

Shiraishi, Kubo, Awata, Odake '95;

- Expand scalar fields in modes

$$\phi^{(a)}(z) = \mathrm{Q}^{(a)} + P^{(a)} \log z + \sum_{k\neq 0} c_k^{(a)} \frac{z^{-k}}{k},$$

$$\left[c_k^{(a)}, c_m^{(b)}\right] = k\delta_{k+m,0}\delta_{a,b}$$

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- *Screening current:*

$$S_{(a)}(x) = \, :\exp\left[\sqrt{\beta}\sum_{k\neq 0}\left(c_k^{(a)} - c_k^{(a+1)}\right)\frac{x^{-k}}{k}\right]:$$

- *Vertex operator:*

$$V_{\vec{\alpha}}(z) = \, :\exp\left[\frac{1}{\sqrt{\beta}}\sum_{k\neq 0}\sum_{a=1}^{n+1} c_k^{(a)}\,\alpha_a\,\frac{z^{-k}}{k}\right]:$$

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- *Screening current:*

$$v = \sqrt{qt^{-1}}$$

$$S^q_{(a)}(x) = :\exp\left(-\sum_{k>0}\frac{1-t^k}{1-q^k} c_k^{(a)}\frac{x^{-k}}{k} + \sum_{k>0} c_{-k}^{(a)}\frac{x^k}{k}\right)\times \exp\left(\sum_{k>0}\frac{1-t^k}{1-q^k} v^k\, c_k^{(a+1)}\frac{x^{-k}}{k} - \sum_{k>0} v^k\, c_{-k}^{(a+1)}\frac{x^k}{k}\right):$$

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- *Vertex operator:*

$$V^q_{\vec{\alpha}}(z) = \ :\exp\left(\sum_{k>0}\sum_{a=1}^{n+1}\frac{q^{k\alpha_a}-1}{1-q^k}c_k^{(a)}v^{-ka}\frac{z^{-k}}{k} + \right.$$
$$\left.\sum_{k>0}\sum_{a=1}^{n+1}\frac{\left(q^{-k\alpha_a}-v^{2k(N-a-1)}\right)}{1-t^k}c_{-k}^{(a)}v^{ka}\frac{z^k}{k}\right):$$

qDF integrals

- Find qDF free field correlators normal ordering q-deformed screening currents and vertex operators. $u = \sqrt{qt}$

$$
q\mathrm{DF}^{A_n}_{l+2} \sim \oint \prod_{a=1}^{n} \prod_{i=1}^{N_a} dx_i^{(a)} \left(x_i^{(a)}\right)^{\beta(N_a - N_{a+1} - 1) + (\alpha_a^{(0)} - \alpha_{a+1}^{(0)})}
$$

$$
\prod_{a=1}^{n} \prod_{i \neq j}^{N_a} \frac{\left(\frac{x_j^{(a)}}{x_i^{(a)}};q\right)_\infty}{\left(t\frac{x_j^{(a)}}{x_i^{(a)}};q\right)_\infty} \prod_{a=1}^{n-1} \prod_{i=1}^{N_a} \prod_{j=1}^{N_{a+1}} \frac{\left(u\frac{x_j^{(a+1)}}{x_i^{(a)}};q\right)_\infty}{\left(v\frac{x_j^{(a+1)}}{x_i^{(a)}};q\right)_\infty} \prod_{p=1}^{l} \prod_{a=1}^{n} \prod_{i=1}^{N_a} \frac{\left(q^{\alpha_{a+1}^{(p)}} v^{-a} \frac{x_i^{(a)}}{z_p};q\right)_\infty}{\left(q^{\alpha_a^{(p)}} v^{-a} \frac{x_i^{(a)}}{z_p};q\right)_\infty}
$$

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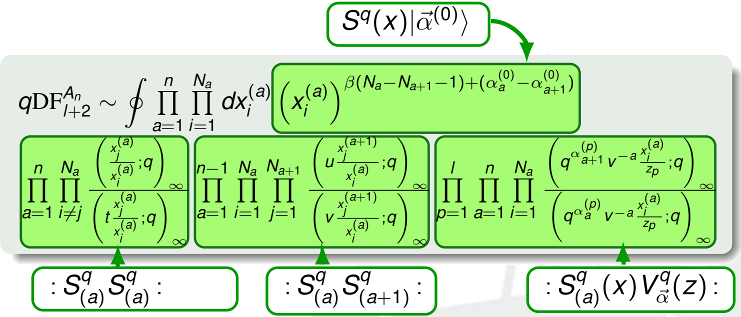

$S^q(x)|\vec{\alpha}^{(0)}\rangle$

$$q\mathrm{DF}^{A_n}_{l+2} \sim \oint \prod_{a=1}^{n}\prod_{i=1}^{N_a} dx_i^{(a)} \boxed{\left(x_i^{(a)}\right)^{\beta(N_a-N_{a+1}-1)+(\alpha_a^{(0)}-\alpha_{a+1}^{(0)})}}$$
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$:S^q_{(a)}S^q_{(a)}:$

$:S^q_{(a)}S^q_{(a+1)}:$

$:S^q_{(a)}(x)V^q_{\vec{\alpha}}(z):$

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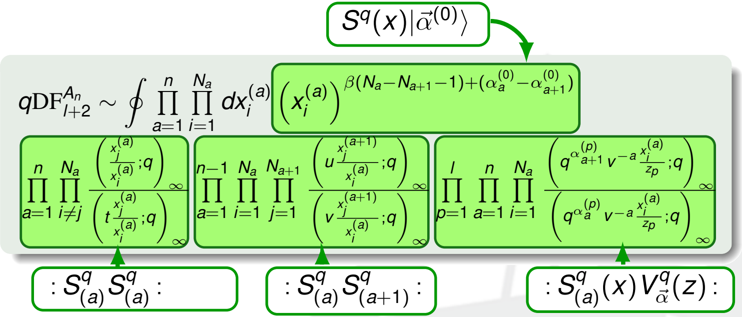

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- This looks exactly like the *partition function of 3d quiver gauge theory on $D^2 \times S^1$.*

  Aganagic, Haouzi, Kozcaz, Shakirov '13; Aganagic, Haouzi, Shakirov '14;

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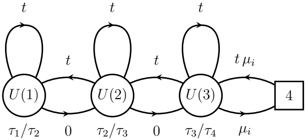
Gauge/CFT dictionary for $FT[SU(N)]$



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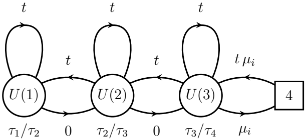

Holomrphic block of $T[SU(N)]$ maps to $(N+2)$-point conformal block of A_{N-1} q-Toda with 2 generic and N degenerate primaries.

$$\boxed{\mathcal{B}^{D_2\times S^1}_{FT[SU(N)]} = q\mathrm{DF}^{A_{N-1}}_{N+2}}$$

$$\vec{\alpha}^{(p)} = \beta\vec{\omega}_{N-1}\,,$$
$$p = 1,\ldots,N$$

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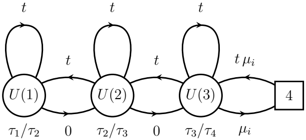

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$$\vec{\alpha}^{(p)} = \beta\vec{\omega}_{N-1}\,,\quad p = 1,\dots,N$$

$\mathcal{B}^{D_2\times S^1}_{T[U(N)]}$	$q\mathrm{DF}^{A_{N-1}}_{N+2}$
Ranks of gauge groups	Screening chrges $N_a = a$
Parameter $q = e^{\hbar} = e^{R\epsilon}$	Deformation parameter q
Axial mass $m_A = t = q^{\beta}$	Central charge parameter β
Vector masses μ_p	Positions of the vertex operators z_p
FI parameters T_a	Initial state momentum $\vec{\alpha}^{(0)}$

$3d$ duality web

$$\begin{tikzcd}
\mathcal{B}^{D_2\times S^1}_{FT[SU(N)]} \arrow[r, leftrightarrow, "\text{spectral dual}"] \arrow[d, leftrightarrow, "\text{gauge}/q\text{DF}"'] & \check{\mathcal{B}}^{D_2\times S^1}_{FT[SU(N)]} \arrow[d, leftrightarrow, "\text{gauge}/q\text{DF}"] \\
q\mathrm{DF}^{A_{N-1}}_{N+2} \arrow[r, leftrightarrow, "\text{spectral dual}"] & q\check{\mathrm{DF}}^{A_{N-1}}_{N+2}
\end{tikzcd}$$

- Horizontal lines ⇔ dualities. Complicated integral identites.
- Vertical lines ⇔ AGT-like correspondences. Simple mapping of parameters.

3*d* duality web

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\mathcal{B}^{D_2\times S^1}_{FT[SU(N)]} \arrow[r, leftrightarrow, "\text{spectral dual}"] \arrow[d, leftrightarrow, "\text{gauge}/q\mathrm{DF}"'] & \check{\mathcal{B}}^{D_2\times S^1}_{FT[SU(N)]} \arrow[d, leftrightarrow, "\text{gauge}/q\mathrm{DF}"] \\
q\mathrm{DF}^{A_{N-1}}_{N+2} \arrow[r, leftrightarrow, "\text{spectral dual}"] & q\check{\mathrm{DF}}^{A_{N-1}}_{N+2}
\end{tikzcd}$$

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- Vertical lines ⇔ AGT-like correspondences. Simple mapping of parameters.

Spectral duality of 3*d* gauge theories ➡ Spectral duality of *q*-Toda conformal blocks

$$q\mathrm{DF}^{A_{N-1}}_{N+2} = q\check{\mathrm{DF}}^{A_{N-1}}_{N+2}$$

Spectral duality exchanges external momentum $\vec{\alpha}^{(0)}$ with the coordinates of vertex operator isnertions z_p

$$\alpha^{(0)}_p \leftrightarrow \log_q z_p$$

Part II:

What happens when we reduce $3d$ web down to $2d$ ($q \to 1$)?

$$\begin{tikzcd}
 & \mathcal{B}^{D^2\times S^1}_{FT[SU(N)]} \arrow[rr, leftrightarrow, "\text{spectral dual}"] \arrow[dl, leftrightarrow] \arrow[dd] & & \check{\mathcal{B}}^{D^2\times S^1}_{FT[SU(N)]} \arrow[dl, leftrightarrow] \arrow[dd] \\
q\mathrm{DF}^{A_{N-1}}_{N+2} \arrow[rr, leftrightarrow, "\text{spectral dual}"] \arrow[dd] & & q\check{\mathrm{DF}}^{A_{N-1}}_{N+2} \arrow[dd] & \\
 & ? \arrow[rr, leftrightarrow] \arrow[dl, leftrightarrow] & & ? \arrow[dl, leftrightarrow] \\
? \arrow[rr, leftrightarrow] & & ? &
\arrow[from=1-2, to=2-3, phantom, "1"]
\arrow[from=3-2, to=4-3, phantom, "2"]
\arrow[from=2-1, to=4-3, phantom, "3"]
\arrow[from=1-2, to=3-4, phantom, "4"]
\arrow[from=1-2, to=4-1, phantom, "5"]
\arrow[from=1-4, to=4-3, phantom, "6"]
\end{tikzcd}$$

General aspects

- Long known result: *3d abelian mirror pair* reduces to *2d Hori-Vafa duals pair*. *Aganagic, Hori, Karch, Tong '01*
- Recent detailed discussion of the topic: *Aharony, Razamat, Willett '17*
  - It is crucial to *turn on all possible mass deformations* keeping vacua of $2d$ theories isolated. Massless limit is very subtle and we avoid it.
  - It is important to define *scaling of 3d mass parameters* with S^1 radius R. Depending on this one can obtain either $2d$ gauge theories or Landau-Ginzburg (LG) theory.
  - One can also obtain *direct sums of 2d theories*.

Higgs limit

Higgs limit:

$$\boxed{q = e^{\hbar} = e^{R\epsilon} \to 1}$$

- $3d$ *FI parameters* T_a scale as $1/R$ and lift Coulomb branch.
- $3d$ *Vector mass* parameters M_a do not scale and matter fields become light.
- *Axial mass* always does not scale.

$$\tau_a = e^{RT_a} \text{ - } \textit{fixed!}; \quad \mu_a = e^{RM_a} = q^{f_a}; \quad m_A = q^{\beta};$$

- *Vacua* are at finite distances

$$x_i^{(a)} = q^{w_i^{(a)}}$$

- Higgs limit is *natural* from gauge theory point of view.

Coulomb limit

Coulomb limit:

$$\boxed{q = e^{\hbar} = e^{R\epsilon} \to 1}$$

- $3d$ *FI parameters* T_a do not scale with R.
- $3d$ *Vector mass* parameters M_a scale as $1/R$ and lift Higgs branch.
- *Axial mass* again does not scale.

$$\tau_a = e^{RT_a} = q^{t_a}; \quad \mu_a = e^{RM_a} \text{ - } \textit{fixed!}; \quad m_A = q^{\beta};$$

- *Vacua* are at infinity

$$x_i^{(a)} \text{ - } \textit{fixed!}$$

- Coulomb limit is *un-natural* from gauge theory point of view.

$FT[SU(N)]$ reduction

- *Higgs limit* of $FT[SU(N)]$ holomorphic block $\mathcal{B}^{D^2\times S^1}_{FT[SU(N)]}$

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3*d* vector mult.

$$\boxed{\prod_{i\neq j}\frac{\left(q^{w_j^{(a)}-w_i^{(a)}};q\right)_\infty}{\left(q^{w_j^{(a)}-w_i^{(a)}+\beta};q\right)_\infty}}$$

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$3d$ vector mult.

?

$$\prod_{i\neq j}\frac{\left(q^{w_j^{(a)}-w_i^{(a)}};q\right)_\infty}{\left(q^{w_j^{(a)}-w_i^{(a)}+\beta};q\right)_\infty} \qquad q\to 1 \qquad \prod_{i\neq j}(-\hbar)^{\beta}\frac{\Gamma\left(w_j^{(a)}-w_i^{(a)}+\beta\right)}{\Gamma\left(w_j^{(a)}-w_i^{(a)}\right)}$$

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$$\boxed{\text{3}d\text{ vector mult.}} \qquad\qquad \boxed{\text{2}d\text{ vector mult.}}$$

$$\boxed{\prod_{i\neq j}\frac{\left(q^{w_j^{(a)}-w_i^{(a)}};q\right)_\infty}{\left(q^{w_j^{(a)}-w_i^{(a)}+\beta};q\right)_\infty}} \Rightarrow q\to 1 \;\boxed{\prod_{i\neq j}(-\hbar)^{\beta}\frac{\Gamma\left(w_j^{(a)}-w_i^{(a)}+\beta\right)}{\Gamma\left(w_j^{(a)}-w_i^{(a)}\right)}}$$

Hori, Romo '13; Honda, Okuda '13;

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$$\overset{\boxed{3d\text{ vector mult.}}}{\boxed{\prod_{i\neq j}\frac{\left(q^{w_j^{(a)}-w_i^{(a)}};q\right)_\infty}{\left(q^{w_j^{(a)}-w_i^{(a)}+\beta};q\right)_\infty}}}\ \underset{q\to 1}{\Rightarrow}\ \overset{\boxed{2d\text{ vector mult.}}}{\boxed{\prod_{i\neq j}(-\hbar)^{\beta}\frac{\Gamma\left(w_j^{(a)}-w_i^{(a)}+\beta\right)}{\Gamma\left(w_j^{(a)}-w_i^{(a)}\right)}}}$$

Hori, Romo '13; Honda, Okuda '13;

- Calculation works similarly for other $3d$ multiplets reducing them to $2d$ multiplets.
- Classical part also reduces to $2d$ one.

$FT[SU(N)]$ reduction

- *Higgs limit* of $FT[SU(N)]$ holomorphic block $\mathcal{B}^{D^2\times S^1}_{FT[SU(N)]}$

3d vector mult.

$$\prod_{i\neq j}\frac{\left(q^{w_j^{(a)}-w_i^{(a)}};q\right)_\infty}{\left(q^{w_j^{(a)}-w_i^{(a)}+\beta};q\right)_\infty}$$

$q\to 1$

2d vector mult.

$$\prod_{i\neq j}(-\hbar)^{\beta}\frac{\Gamma\left(w_j^{(a)}-w_i^{(a)}+\beta\right)}{\Gamma\left(w_j^{(a)}-w_i^{(a)}\right)}$$

Hori, Romo '13; Honda, Okuda '13;

- Calculation works similarly for other 3d multiplets reducing them to 2d multiplets.
- Classical part also reduces to 2d one.
- Summarizing we find 3d holomorphic block reduces to 2d block of the same theory

$$\lim_{q\to 1}\mathcal{B}^{D_2\times S^1}_{FT[SU(N)]}(\vec{\tau},\vec{\mu},t)=(-\hbar)^{-\beta\frac{N(N-1)}{2}}\mathcal{B}^{D_2}_{FT[SU(N)]}\left(\vec{\tau},\vec{f},\beta\right)$$

Dual $FT[SU(N)]$ reduction

- For dual the dual $FT[SU(N)]$ theory Higgs limit turns into *Coulomb limit*.

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$$\overset{\boxed{3d\text{ vector mult.}}}{\boxed{\prod_{i\neq j}\frac{\left(x_j^{(a)}/x_i^{(a)};q\right)_\infty}{\left(q^\beta x_j^{(a)}/x_i^{(a)};q\right)_\infty}}} \qquad q\to 1 \qquad \overset{\boxed{?}}{\boxed{\prod_{i\neq j}\left(1-\frac{x_j^{(a)}}{x_i^{(a)}}\right)^\beta}}$$

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$$\overset{\boxed{3d\text{ vector mult.}}}{\boxed{\prod_{i\neq j}\frac{\left(x_j^{(a)}/x_i^{(a)};q\right)_\infty}{\left(q^\beta x_j^{(a)}/x_i^{(a)};q\right)_\infty}}} \quad \overset{\Rightarrow}{q\to 1} \quad \overset{\boxed{:s_{(a)}s_{(a)}:}}{\boxed{\prod_{i\neq j}\left(1-\frac{x_j^{(a)}}{x_i^{(a)}}\right)^\beta}}$$

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3d vector mult.

$: s_{(a)} s_{(a)} :$

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- Similar form of the limit arises for other multiplets.
- Summing up all contributions we obtain Selberg-like integral.

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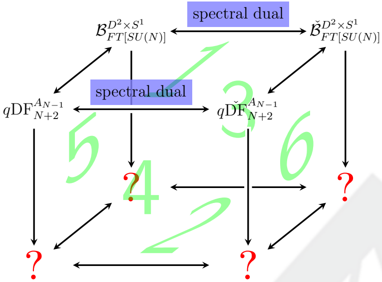
- Similar form of the limit arises for other multiplets.
- Summing up all contributions we obtain Selberg-like integral.
- We interpret this integral as Conformal block of Toda CFT

$$\lim_{q\to 1}\check{\mathcal{B}}^{D_2\times S^1}_{FT[SU(N)]}(\vec{\tau},\vec{\mu},t)=(-\hbar)^{-\beta\frac{N(N-1)}{2}}\check{\mathrm{DF}}^{A_{N-1}}_{N+2}$$

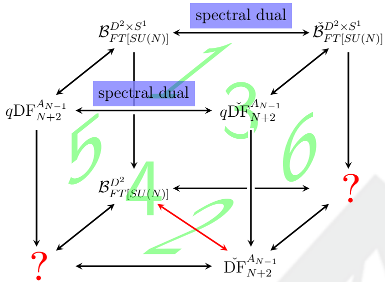
- Notice exactly the same power of $\hbar$ as in the Higgs limit!

Reduction of 3d duality web



$$\begin{tikzcd}
& \mathcal{B}^{D^2\times S^1}_{FT[SU(N)]} \arrow[rr, leftrightarrow, "\text{spectral dual}"] \arrow[dl, leftrightarrow] \arrow[dd] & & \check{\mathcal{B}}^{D^2\times S^1}_{FT[SU(N)]} \arrow[dl, leftrightarrow] \arrow[dd] \\
q\mathrm{DF}^{A_{N-1}}_{N+2} \arrow[rr, leftrightarrow, "\text{spectral dual}"] \arrow[dd] & & q\check{\mathrm{DF}}^{A_{N-1}}_{N+2} \arrow[dd] & \\
& ? \arrow[rr, leftrightarrow] \arrow[dl, leftrightarrow] & & ? \arrow[dl, leftrightarrow] \\
? \arrow[rr, leftrightarrow] & & ? &
\end{tikzcd}$$

Reduction of $3d$ duality web



- The red arrow is the *$2d$ gauge/CFT correspondence* between vortex partition functions and Toda degenerate correlators.

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Reduction of $3d$ duality web

$$\begin{tikzcd}
 & \mathcal{B}^{D^2\times S^1}_{FT[SU(N)]} \arrow[rr, leftrightarrow, "\text{spectral dual}"] \arrow[dl, leftrightarrow] \arrow[dd] & & \check{\mathcal{B}}^{D^2\times S^1}_{FT[SU(N)]} \arrow[dl, leftrightarrow] \arrow[dd] \\
q\mathrm{DF}^{A_{N-1}}_{N+2} \arrow[rr, leftrightarrow, "\text{spectral dual}"] \arrow[dd] & & q\check{\mathrm{DF}}^{A_{N-1}}_{N+2} \arrow[dd] & \\
 & \mathcal{B}^{D^2}_{FT[SU(N)]} \arrow[rr, leftrightarrow, "2d\ \text{massive dual}"] \arrow[dr, leftrightarrow, red] & & \mathcal{B}^{D^2}_{\mathrm{LG}} \\
? \arrow[ur, leftrightarrow] \arrow[rr, leftrightarrow] & & \check{\mathrm{DF}}^{A_{N-1}}_{N+2} \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

- The red arrow is the *2d gauge/CFT correspondence* between vortex partition functions and Toda degenerate correlators.

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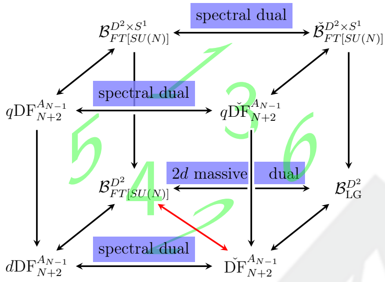
- $\mathcal{B}^{D_2}_{LG}$ is the D^2 partition function of a *theory of twisted chiral multiplets with chiral superpotential*. *Aharony, Razamat , Willett'17*

Reduction of $3d$ duality web



- The red arrow is the *2d gauge/CFT correspondence* between vortex partition functions and Toda degenerate correlators.

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- $\mathcal{B}^{D_2}_{LG}$ is the D^2 partition function of a *theory of twisted chiral multiplets with chiral superpotential*. *Aharony, Razamat , Willett'17*
- $d\mathrm{DF}^{A_{N-1}}_{N+2}$ is a correlator of vertex operators in *theory with $d\mathcal{W}_N$ current algebra*, an *un-natural limit* of $q\mathcal{W}_N$.

Bonus: *d*-Virasoro algebra

- Start from the $\mathcal{V}ir_{q,t}$ commutation relation. $T(z) = \sum_{n\in\mathbb{Z}} T_n z^n$

$$f\left(\tfrac{w}{z}\right) T(z)T(w) - f\left(\tfrac{z}{w}\right) T(w)T(z) = -\frac{(1-q)(1-t^{-1})}{1-\frac{q}{t}} \left(\delta\left(\tfrac{q}{t}\tfrac{w}{z}\right) - \delta_\times\left(\tfrac{t}{q}\tfrac{w}{z}\right)\right)$$

- Take *Higgs-like limit* $z = q^u,\ w = q^v$ *(un-natural from qCFT point of view)*

$$g(v-u)t(u)t(v) - g(u-v)t(v)t(u) = -\frac{\beta}{1-\beta}\left(\delta(v-u+1-\beta) - \delta(v-u-1+\beta)\right)$$

$$t(u) \equiv \lim_{q\to 1} T\left(q^u\right) \qquad g(u) = \frac{2(1-\beta)}{u}\frac{\Gamma\left(\frac{u+2-\beta}{2(1-\beta)}\right)\Gamma\left(\frac{u+1-2\beta}{2(1-\beta)}\right)}{\Gamma\left(\frac{u+1}{2(1-\beta)}\right)\Gamma\left(\frac{u-\beta}{2(1-\beta)}\right)}$$

- *Bosonize* this algebra: find screening currents and vertex operators.
- Derive dDF conformal blocks

Outlook

- Generalize to other dual pairs. First step - abelian quivres. Work in progress with *Aprile, Pasquetti, Sacchi and Zenkevich*
- Fix subtleties, e.g. mapping of integration contours, boundary conditions on D^2.
- Work out $2d$ gauge/CFT correspondence without relying on $3d$ mirror symmetry.
- Understand better relation between LG theories and DF representations of conformal blocks.
- Study d-Virasoro algebra and related issues (conformal blocks, integrable hamiltonians etc.)

Thank you!