$T[SU(N)]$ duality webs

Sara Pasquetti

Milano-Bicocca University

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based on arXiv:1712.08140 with Nedelin and Zenkevich, arXiv:1808.XXXX with Aprile and work in progress with Aprile, Nedelin, Sacchi and Zenkevich

Plan

- $T[SU(N)]$ duality web and its deformations
- 3d spectral dualities from 5d fiber-base dualities via Higgsing
- spectral dualities and gauge/q-CFT correspondence

3d $\mathcal{N} = 2$ basics

Field content:

- Vector multiplets: $V = (A_\mu, \lambda, \sigma \in \mathbb{R}, aux)$
- Adjoint chiral multiplets: $\Phi = (\phi \in \mathbb{C}, fermions, aux)$
- Matter chiral multiplets: $Q_i = (Q_i \in \mathbb{C}, fermions, aux)$
- One can also introduce the linear multiplets: $\Sigma = (\sigma, .., F_{\mu\nu})$.

The moduli space of vacua contains the pure Higgs branch where $< Q_i > \neq 0$ and $< \sigma >= 0$, Coulomb branch where σ gets a vev which breaks the gauge group to its Cartan: $G \to U(1)^r$, Mixed branches.

In the bulk of the (abelianised) Coulomb branch one can dualise the gauge fields to scalars: $F^j_{\mu\nu} = \epsilon_{\mu\nu\rho}\partial^\rho \gamma_j\,,\ j = 1, \cdots r.$

The currents $J^j_\mu = \epsilon_{\mu\nu\rho}(F^{\nu\rho})^j$ generate the topological symmetry $(U(1)_J)^r$ which shifts the dual photons γ_j.

A set of convenient coordinates on the classical Coulomb branch are:

$$X_j \sim e^{\Phi_j}\,, \quad \Phi_j = \frac{2\pi\sigma_j}{g_3^2} + i\gamma_j\,, \quad j = 1, \cdots, r$$

Quantum corrections can lift the Coulomb branch. For $U(N_c)$ with $N_f > N_c$ only $X_+ \sim e^{(\frac{\pi\sigma_1}{g_3^2} + i\gamma_1)}, X_- \sim e^{-(\frac{\pi\sigma_{N_c}}{g_3^2} + i\gamma_{N_c})}$ survive.

The un-lifted coordinates are identified with half BPS monopoles, local disorder operators. Their charges under any Abelian symmetry is computed by

$$\delta Q(\mathfrak{M}) = -\frac{1}{2}\sum_{\text{fermions } \psi} Q(\psi)\left|\rho_\psi(\mathfrak{m})\right|\,.$$

where the fermions ψ transform with ρ_ψ under the gauge group.

Aharony-like dualities & monopole deformations

- Aharony duality:

  $\mathcal{T}$: $U(N_c)$ with N_f flav. $Q, \tilde{Q}$, $\mathcal{W} = 0$

  $\mathcal{T}'$: $U(N_f - N_c)$ with N_f flav. $q, \tilde{q}$, $\mathcal{W} = S_- \hat{\mathfrak{M}}^+ + S_+ \hat{\mathfrak{M}}^- + Mq\tilde{q}$

- Monopole duality I: [Benini-Benvenuti-SP]

  $\mathcal{T}_{\mathfrak{M}}$: $U(N_c)$ with N_f flav. $Q, \tilde{Q}$, $\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^-$

  $\mathcal{T}'_{\mathfrak{M}}$: $U(N_f - N_c - 2)$ with N_f flavors $q, \tilde{q}$, $\mathcal{W} = \hat{\mathfrak{M}}^+ + \hat{\mathfrak{M}}^- + Mq\tilde{q}$

- Monopole duality II: [Benini-Benvenuti-SP]

  $\mathcal{T}_{\mathfrak{M}}$: $U(N_c)$ with N_f flav. $Q, \tilde{Q}$, $\mathcal{W} = \mathfrak{M}^+$

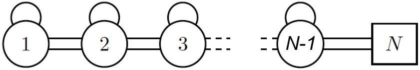
  $\mathcal{T}'_{\mathfrak{M}}$: $U(N_f - N_c - 1)$ with N_f flav. $q, \tilde{q}$, $\mathcal{W} = \hat{\mathfrak{M}}^- + S_+ \hat{\mathfrak{M}}^+ + Mq\tilde{q}$.

Monopole super-potentials naturally appear in 4d-3d reductions [Aharony-Razamat-Seibger-Willett].

Generalizations to $UsP(N), O(N)$ groups, higher monopole deformations [Amariti-Garozzo-Mekareeya], quivers [Amariti-Orlando-Reffert], $\cdots$

$T[SU(N)]$

The $\mathcal{N}=4$ $T[SU(N)]$ theory [Gaiotto-Witten] is a quiver theory



with $\mathcal{W}_{T[SU(N)]} = \sum_{k=1}^{N-1} \mathrm{Tr}_k \left[\Phi_k \left(\mathrm{Tr}_{k+1}\mathbb{Q}^{(k,k+1)} - \mathrm{Tr}_{k-1}\mathbb{Q}^{(k-1,k)}\right)\right]$,
with bifund. $\mathbb{Q}^{(L,R)} = Q_{ab}^{(L,R)}\tilde{Q}_{\tilde{a}\tilde{b}}^{(L,R)}$.

- Global symmetry: $SU(N)_F \times SU(N)_{top}$
- Self-dual under mirror symmetry: Coulomb $\leftrightarrow$ Higgs branch
- Real masses M_p, T_p in $SU(N)_F \times SU(N)_{top}$
- Real axial mass $m_A \in SU(2)_C \times SU(2)_H$ breaking to $\mathcal{N}=2^*$
- The mass deformed theory has $N!$ isolated vacua

$T[SU(N)]$ and its mirror dual $T[SU(N)]^V$

The chiral ring generators are the mesons on the Higgs branch:

$$\mathcal{Q}_{ij} \equiv \mathrm{Tr}_N \mathbb{Q}^{(N-1,N)} \equiv Q_i \tilde{Q}_j\,, \qquad R[\mathcal{Q}_{ij}] = 2r$$

and the monopole operators on the Coulomb branch:

$$\mathcal{M}_{ij} \equiv \begin{pmatrix} \mathrm{Tr}\Phi^{(1)} & \mathcal{M}^{100} & \mathcal{M}^{110} & \mathcal{M}^{111} \\ \mathcal{M}^{-100} & \mathrm{Tr}\Phi^{(2)} & \mathcal{M}^{010} & \mathcal{M}^{011} \\ \dots & \dots & \dots & \dots \end{pmatrix}, \qquad R[\mathcal{M}_{ij}] = 2-2r\,.$$

In the mirror (self)-dual theory $T[SU(N)]^V$ we have dual mesons on the Higgs branch with $R[\mathcal{P}_{ij}] = 2-2r$ and the monopole matrix with $R[\mathcal{N}_{ij}] = 2r$ on the Coulomb branch.

Operator map:

$$\mathcal{Q}_{ij} \leftrightarrow \mathcal{N}_{ij}\,, \qquad \mathcal{M}_{ij} \leftrightarrow \mathcal{P}_{ij}$$

The partition functions must satisfy:

$$Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A) = Z_{T[SU(N)]}(\vec{T}, \vec{M}, -m_A)\,.$$

Difference operators

The $T[SU(N)]$ partition function is an eigenfunction of the trigonometric Ruijsenaars-Schneider (RS) Hamiltonians [Gaiotto-Koroteev],[Bullimore-Kim-Koroteev]:

$$T_r(\vec{M}, m_a) Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A) = \chi_r(\vec{T}) Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A)$$

$$T_r(\vec{T}, -m_a) Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A) = \chi_r(\vec{M}) Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A)\,,$$

with $r = 1, \cdots, N$, implying the identity for mirror self-duality:

$$Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A) = Z_{T[SU(N)]}(\vec{T}, \vec{M}, -m_A)\,.$$

Moreover since $T_r(\vec{M}, -m_a) = K[\vec{M}, m_A]^{-1} T_r(\vec{M}, m_a) K[\vec{M}, m_A]$, we get another identity:

$$Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A) = K[\vec{M}, m_A]^{-1} K[\vec{T}, m_A] Z_{T[SU(N)]}(\vec{M}, \vec{T}, -m_A)\,,$$

where $K[\vec{M}, m_A]$ is the contribution of N^2 chirals in the $SU(N)$ adjoint.

This suggests that we have a new duality!

$T[SU(N)]$ and its flip-flip dual $FFT[SU(N)]$

The electric theory is $T[SU(N)]$, the magnetic theory is $FFT[SU(N)]$, the same quiver theory where we *flip* the mesons $R[q_i \tilde{q}_j] = 2r' = 2 - 2r$ and the monopoles $R[\mathrm{m}_{ij}] = 2 - 2r' = 2r$:

$$\mathcal{W}_{FFT[SU(N)]} = \mathcal{W}_{T[SU(N)]} + S_{ij}\mathrm{m}_{ij} + q_i \tilde{q}_j X_{ij}.$$

Now the moduli space is parameterized by the flipping fields with:

$$R[X_{ij}] = 2r\,, \qquad R[S_{ij}] = 2 - 2r.$$

and the operator map:

$$\mathcal{Q}_{ij} \leftrightarrow X_{ij}\,, \qquad \mathcal{M}_{ij} \leftrightarrow S_{ij}\,.$$

This is a generalized Aharony duality!

$T[SU(N)]^V$ and its flip-flip dual $FFT[SU(N)]^V$

Similarly on the mirror side we have a duality between $T[SU(N)]^V$ and $FFT[SU(N)]^V$, the same quiver theory where we *flip* the mesons $R[p_i\tilde{p}_j] = 2r'' = 2 - 2r' = 2r$ and the monopoles $R[\mathrm{n}_{ij}] = 2 - 2r$, with:

$$\mathcal{W}_{FFT[SU(N)]^V} = \mathcal{W}_{T[SU(N)]} + \mathrm{n}_{ij} R_{ij} + p_i \tilde{p}_j Y_{ij} \, .$$

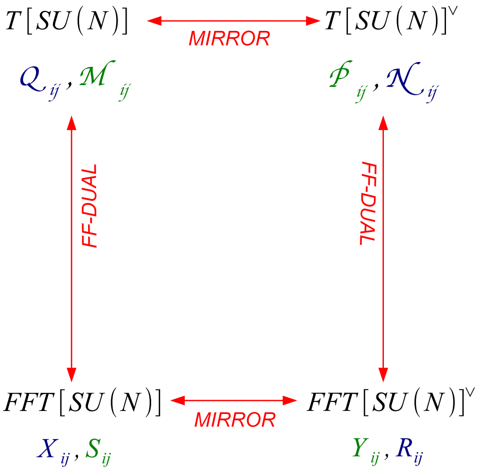
Now the moduli space is parameterised by the flipping fields with:

$$R[Y_{ij}] = 2 - 2r \, , \qquad R[R_{ij}] = 2r \, ,$$

and the operator map is:

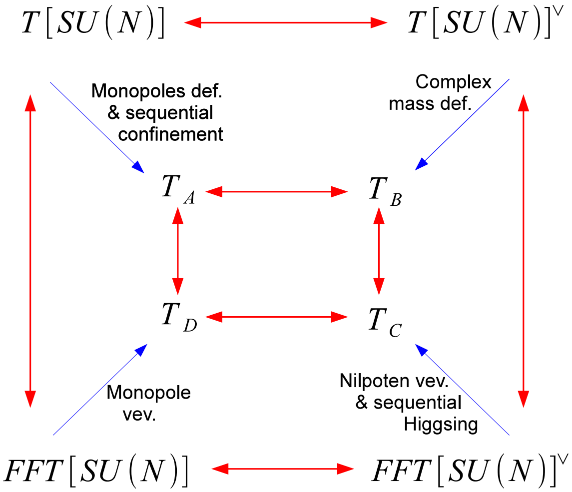
$$\mathcal{P}_{ij} \to Y_{ij} \, , \qquad \mathcal{N}_{ij} \to R_{ij} \, .$$

$T[SU(N)]$ duality web



Deforming the $T[SU(N)]$ duality web

The duality web can be deformed to generate new webs. We monopole deform $T[SU(N)]$ and follow what happens in the various frames:



$T[SU(N)] \to \mathcal{T}_A$: sequential confinement

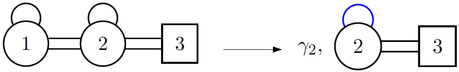
As in [Benvenuti-Giacomelli],[Giacomelli-Mekareeya] we *sequentially confine* $T[SU(N)]$ by turning on

$$\delta\mathcal{W} = \mathcal{M}^{10\cdots 0} + \mathcal{M}^{01\cdots 0} + \cdots + \mathcal{M}^{0\cdots 10}\,,$$

using at each node the duality [Benini-Benvenuti-SP]:

$$U(N), \quad N_f = N+1 \text{ with } \mathcal{W} = \mathcal{M}^{+} \leftrightarrow \text{WZ with } \mathcal{W} = \gamma \det M\,.$$

Example in $T[SU(3)]$ we turn on $\delta\mathcal{W} = \mathcal{M}^{10}$. Since the adjoint in the first node is decoupled we can apply the monopole duality:



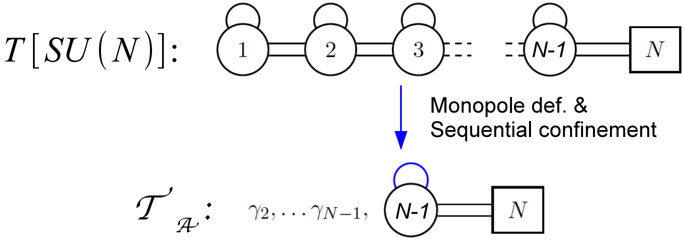
After integrating out massive fields, we find at low energy

$$\mathcal{W} = \tfrac{\phi_2}{2} Tr[\mathcal{Q}] - \tfrac{\gamma_2}{2} Tr[\mathcal{Q}^2]$$

Notice that the adjoint is abelian.

$T[SU(N)] \to \mathcal{T}_A$: sequential confinement

Iterating this procedure we see that in the monopole deformed $T[SU(N)]$ all the nodes but the last one are confined:



The final theory $\mathcal{T}_A$ has an abelian adjoint ϕ and $N-2$ singlets γ_m, with:

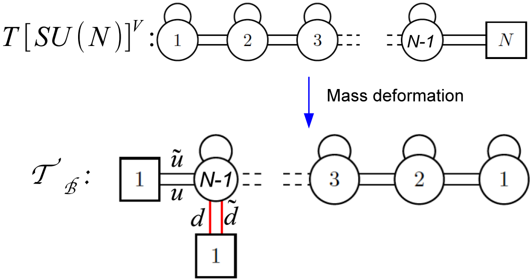
$$\mathcal{W}_A = \frac{1}{(N-1)!}\phi\, Tr[\mathcal{Q}] - \sum_{m=2}^{N-1} \frac{\gamma_m}{m} Tr[\mathcal{Q}^m]\,.$$

$T[SU(N)]^V \to \mathcal{T}_B$: mass deformation

On the mirror side the monopole deformation is mapped to a complex mass deformation:

$$\delta\mathcal{W} = P_1\tilde{P}_2 + P_2\tilde{P}_3 + \cdots + P_{N-2}\tilde{P}_{N-1}.$$

Only two flavors $(P_1, \tilde{P}_{N-1}) \equiv (d, \tilde{d})$ and $(P_N, \tilde{P}_N) \equiv (u, \tilde{u})$ remain light.



with

$$\mathcal{W}_B = d\left(\Phi_{N-1}\right)^{N-1}\tilde{d} + u\Phi_{N-1}\tilde{u} + \mathcal{W}_{tail}\,.$$

$FFT[SU(N)]^V \to \mathcal{T}_C$: nilpotent vev & Higgsing

In $FFT[SU(N)]^V$ with $\mathcal{W}_{FFT[SU(N)]^V} = \mathcal{W}_{T[SU(N)]} + \mathrm{n}_{ij} R_{ij} + p_i \tilde{p}_j Y_{ij}$ the monopole deformation maps to

$$\delta\mathcal{W} = Y_{12} + Y_{23} + \cdots + Y_{N-2,N}$$

The F-terms of the Y_{ij} singlets give a vev to the meson:

$$\langle p_i \tilde{p}_j \rangle = \langle \mathrm{Tr}_{N-1} \mathbb{P}^{(N-1,N)} \rangle = \mathbb{J}_{N-1} \oplus \mathbb{J}_1$$

The F-terms of the adjoints Φ_k, $k = N, \ldots 2$ then propagate the vev:

$$\mathrm{Tr}_{k-1} \mathbb{P}^{(k-1,k)} = \mathrm{Tr}_{k+1} \mathbb{P}^{(k,k+1)} \,,$$

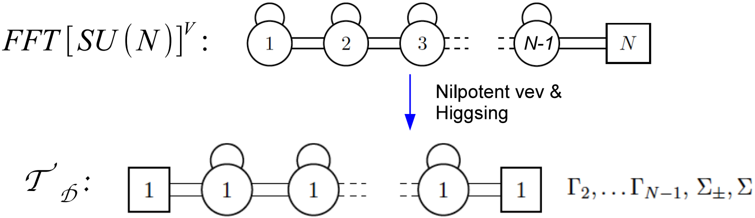
which can be solved for nilpotent vevs for the bifundamental fields. Finally the F-terms for bifundamentals and D-terms give:

$$\langle \Phi_k \rangle = \mathbb{J}_1 \oplus \mathbb{J}_{k-1} \,.$$

$FFT[SU(N)]^V \to \mathcal{T}_C$: nilpotent vev & Higgsing

All these vevs determine a super-Higgs mechanism as in [Agarwal-Bah-Maruyoshi-Song] which has the effect of abelianising all the nodes.

A careful analysis of the mass matrix allow us to find the remaining light fields in the low energy theory, the abelian quiver $\mathcal{T}_C$:



Alternative path: $\mathcal{T}_A \to \mathcal{T}_D \to \mathcal{T}_C$

Starting from $\mathcal{T}_A$, the $U(N-1)$ with N flavors with

$$\mathcal{W}_A = \tfrac{1}{(N-1)!}\phi\, Tr[\mathcal{Q}] - \sum_{m=2}^{N-1} \tfrac{\gamma_m}{m} Tr[\mathcal{Q}^m]\,,$$

we take the Aharony dual (and map the superpotential) and obtain $\mathcal{T}_D$: $U(1)$ with N flavor and

$$\mathcal{W}_D = \mathcal{M}^{\pm} S_{\pm} + Q_i \tilde{Q}_j \hat{Z}_{ij} + \sum_{m=2}^{N} \tfrac{\Gamma_m}{m} Tr[\hat{Z}_{ij}^m]\,.$$

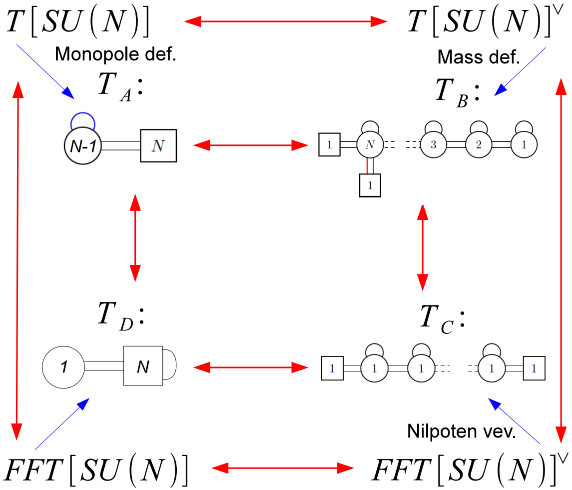
Now we take the mirror (and map the superpotential), after some rearrangement we obtain the abelian quiver theory $\mathcal{T}_C$ with

$$\mathcal{W}_C = \Psi_1(b_1\tilde{b}_1 - b_2\tilde{b}_2) + \cdots + \Psi_{N-1}(b_{N-1}\tilde{b}_{N-1} - b_N\tilde{b}_N) + \Sigma\sum_i b_i\tilde{b}_i +$$

$$+\prod_i \tilde{b}_i\Sigma_+ + \prod_i b_i\Sigma_- + \mathcal{M}_{ij}\hat{Z}_{ij} + \sum_{m=2}^{N-1} \tfrac{\Gamma_m}{m} Tr[\hat{Z}_{ij}^m]\,,$$

consistent with what we got via the nilpotent Higgsing of $FFT[SU(N)]^V$!

Deformed $T[SU(N)]$ duality web



→ More general deformations will lead to new duality webs.

Spectral duality $FT[SU(N)] \leftrightarrow FT[SU(N)]^V$

Starting from the dual pair on the diagonal of the undeformed web we *Flip* on both sides, (since $Flip^2 = 1$) we find a new *spectral* self-dual pair:

$$FFT[SU(N)] \longleftrightarrow T[SU(N)]^V$$

$\delta W = S_{ij} T_{ij}$ ↓ *Flip*   *Flip* ↓ $\delta W = P_{ij} T_{ij}$

$$FT[SU(N)] \longleftrightarrow FT[SU(N)]^V$$

$$W_{FT[SU(N)]} = W_{T[SU(N)]} + Q_{ij} X_{ij} \qquad W_{FT[SU(N)]^V} = W_{T[SU(N)]} + P_{ij} T_{ij}$$

Spectral duality $FT[SU(N)] \leftrightarrow FT[SU(N)]^V$

Operators map:

- Electric side: $FT[SU(N)]$, with $R[\mathcal{Q}_{ij}] = 2r$ we have the monopoles $R[\mathcal{M}_{ij}] = 2 - 2r$ and the singlets $R[X_{ij}] = 2 - 2r$.
- Magnetic side: $FT[SU(N)]^V$, with $R[\mathcal{P}_{ij}] = 2r' = 2r$ we have the monopoles $R[\mathcal{N}_{ij}] = 2 - 2r$ and the singlets $R[T_{ij}] = 2 - 2r$.

$$X_{ij} \leftrightarrow \mathcal{N}_{ij}\,, \qquad \mathcal{M}_{ij} \leftrightarrow T_{ij}$$

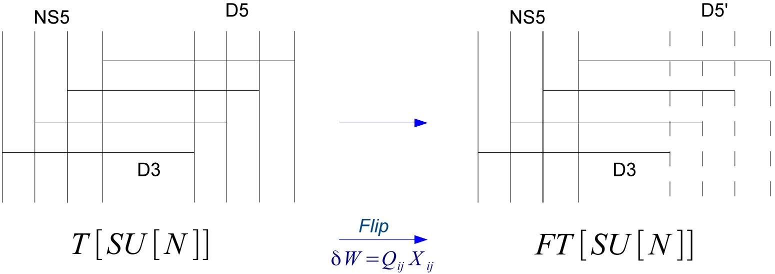
Using the difference operators it easy to check that:

$$Z_{FT[SU(N)]}(\vec{M}, \vec{T}, m_A) = Z_{FT[SU(N)]}(\vec{T}, \vec{M}, m_A)$$

where

$$Z_{FT[SU(N)]}(\vec{M}, \vec{T}, m_A) = K[\vec{M}, m_A] Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A)\,.$$

Brane set-ups



NS5 and D5' branes can form a pq-web engineering a 5d $\mathcal{N} = 1$ theory. We are going to interprete $FT[SU(N)]$ as a codimension-two defect theory and show that spectral duality follows from 5d fiber base duality.

In the following it will be useful to work with $D_2 \times S^1$ partition functions, the holomorphic blocks. So I will quickly introduce them for $T[SU(N)]$.

$T[SU(N)]$ holomorphic block integral

We consider the $D_2 \times S^1$ partition function, or holomorphic block with $q = e^{\hbar}$, $\hbar = R\epsilon$:

$$\mathcal{B}^{D_2 \times S^1}_{T[SU(N)]}(\vec{\mu}, \vec{\tau}, t) = \int \prod_{a=1}^{N-1} \prod_{i=1}^{a} \frac{dx_i^{(a)}}{x_i^{(a)}} Z_{\rm cl}(\vec{\tau})\ Z_{1loop}(\vec{\mu})$$

where $Z_{\rm cl}$ contains all mixed Chern-Simons couplings and

$$Z_{1loop} = \prod_{a=1}^{N-1} \frac{\prod_{i \neq j}^{a} \left(\frac{x_j^{(a)}}{x_i^{(a)}}; q \right)_\infty}{\prod_{i,j=1}^{a} \left(t \frac{x_j^{(a)}}{x_i^{(a)}}; q \right)_\infty} \prod_{a=1}^{N-2} \prod_{i=1}^{a} \prod_{j=1}^{a+1} \frac{\left(t \frac{x_j^{(a+1)}}{x_i^{(a)}}; q \right)_\infty}{\left(\frac{x_j^{(a+1)}}{x_i^{(a)}}; q \right)_\infty} \prod_{p=1}^{N} \prod_{i=1}^{N-1} \frac{\left(t \frac{\mu_p}{x_i^{(N-1)}}; q \right)_\infty}{\left(\frac{\mu_p}{x_i^{(N-1)}}; q \right)_\infty},$$

where $(x;\, q)_\infty = \prod_{k=0}^{\infty} \left(1 - xq^k\right)$ and $\mu_p = e^{RM_p}$, $\tau_p = e^{RT_p}$, $t = e^{Rm_A}$.

The integral is evaluated on a basis of contours Γ_α, $\alpha = 1, \cdots, N!$ in one to one correspondence with the SUSY vacua.

The integration over the reference contour Γ_{α_0} yields

$$\mathcal{B}^{D_2\times S^1,(\alpha_0)}_{T[SU(N)]} = Z^{3d,(\alpha_0)}_{\mathrm{cl}} Z^{3d,(\alpha_0)}_{1loop} Z^{3d,(\alpha_0)}_{\mathrm{vort}} \,,$$

with

$$Z^{3d,(\alpha_0)}_{\mathrm{vort}}(\vec{\mu},\vec{\tau},q,t) =$$

$$= \sum_{\{k_i^{(a)}\}} \prod_{a=1}^{N-1} \left[\left(t\frac{\tau_a}{\tau_{a+1}}\right)^{\sum_{i=1}^a k_i^{(a)}} \prod_{i\neq j}^{a} \frac{\left(t\frac{\mu_i}{\mu_j};q\right)_{k_i^{(a)}-k_j^{(a)}}}{\left(\frac{\mu_i}{\mu_j};q\right)_{k_i^{(a)}-k_j^{(a)}}} \prod_{i=1}^{a}\prod_{j=1}^{a+1} \frac{\left(\frac{q}{t}\frac{\mu_i}{\mu_j};q\right)_{k_i^{(a)}-k_j^{(a+1)}}}{\left(q\frac{\mu_i}{\mu_j};q\right)_{k_i^{(a)}-k_j^{(a+1)}}} \right]$$

the sum is over sets of integers $k_i^{(a)}$ satisfying the inequalities

$$\begin{array}{ccccc} k_1^{(1)} \geq & k_1^{(2)} \geq & k_1^{(3)} \geq & \cdots \geq & k_1^{(N-1)} \geq 0 \\ & k_2^{(2)} \geq & k_2^{(3)} \geq & \cdots \geq & k_2^{(N-1)} \geq 0 \\ & & & \ddots & \vdots \\ & & & & k_{N-1}^{(N-1)} \geq 0 \end{array}$$

Duality identities for the blocks:

$$\text{mirror}: \qquad \mathcal{B}^{D_2\times S^1}_{T[SU(N)]}(\vec{\mu},\vec{\tau},t) = \mathcal{B}^{D_2\times S^1}_{T[SU(N)]}(\vec{\tau},\vec{\mu},\frac{q}{t})\,,$$

$$\text{spectral}: \qquad \mathcal{B}^{D_2\times S^1}_{FT[SU(N)]}(\vec{\mu},\vec{\tau},t) = \mathcal{B}^{D_2\times S^1}_{FT[SU(N)]}(\vec{\tau},\vec{\mu},t)\,.$$

$3d$ $FT[SU(N)]$ and its dual from $5d$

$FT[SU(N)]$ lives on D3 branes suspended between N NS5s and N D5's. These branes form the pq-web engineering the $5d$ $\mathcal{N} = 1$ quiver theory $N + SU(N)^{N-1} + N$.

We want to view $FT[SU(N)]$ as a codimension-two defect in this theory:

- Higgsing: the $FT[U(N)]$ partition function is obtained by tuning the parameters of the $5d$ square quiver partition function.
- Brane realisation: the codimension-two defect theory is the vortex string theory on the Higgs branch of the $5d$ theory.
- Geometric engineering: Higgsing corresponds to geometric transition happening at quantised values of the Kähler parameters.
- 3d spectral duality descends from fiber-base or IIB S-duality

Higgsing the $5d$ square quiver

The instanton partition function $Z^{5d}_{inst}[U(N)^{N-1}]$ is a sum over N-tuple of Young diagrams, $\vec{Y}^{(a)} = \{Y^{(a)}_1, \ldots, Y^{(a)}_N\}$, $a = 1, \ldots, (N-1)$.

When the Coulomb branch parameters are tuned to special values, the Young diagrams for some nodes truncate to diagrams with finitely many columns yielding the partition function of a coupled system:

$$Z^{5d}[U(N)^{N-1}] \xrightarrow[\text{Higgsing}]{} Z^{5d-3d} \,.$$

For *maximal* Higgsing the $5d$ bulk theory is trivial and we just get the vortex partition function of the $3d$ theory.

$FT[SU(N)]$ via Higgsing

By *maximally* Higgsing the $5d$ square quiver by tuning masses and Coulomb parameters as:

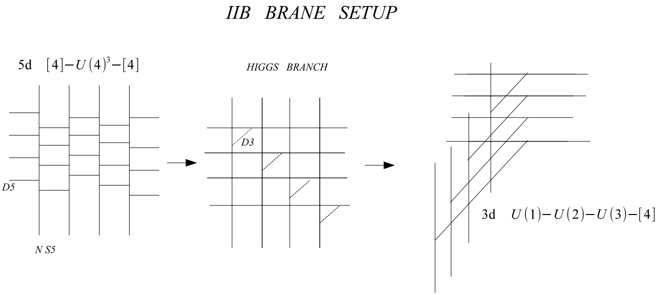
$$\begin{array}{ccccc} a_1^{(1)} = m_1 t, & & a_1^{(N-1)} = m_1 t, & & \bar{m}_1 = m_1 \frac{t^2}{q}, \\ a_2^{(1)} = m_2, & & a_2^{(N-1)} = m_2 t, & & \bar{m}_2 = m_2 \frac{t^2}{q}, \\ \vdots & \cdots & \vdots & & \vdots \\ a_{N-1}^{(1)} = m_{N-1}, & & a_{N-1}^{(N-1)} = m_{N-1} t, & & \bar{m}_{N-1} = m_{N-1} \frac{t^2}{q}, \\ a_N^{(1)} = m_N, & & a_N^{(N-1)} = m_N, & & \bar{m}_N = m_N \frac{t^2}{q} \end{array}$$

we obtain $FT[SU(N)]$:

$$Z^{5d}[U(N)^{N-1}] \to \mathcal{B}_{FT[SU(N)]}^{D_2 \times S^1, (\alpha_0)} .$$

Higgsing and branes

The 5d square quiver can be realised as the low energy description of a web of N NS5 and N D5' branes.



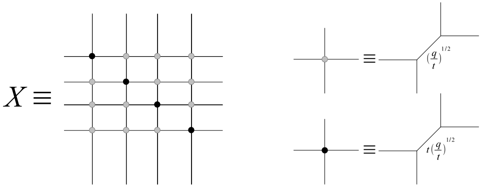
On the Higgs branch the NS5 branes can be removed from the web and D3 stretched in between. The $3d$ low energy theory on the D3s is our vortex theory [Hanany-Tong],[Dorey-Lee-Hollowood].

Higgsing and geometric transition

We can engineer the 5d quiver theory from M theory on $X \times \mathbb{R}^4_{q,t} \times S^1$.

The Higgsing conditions translate into quantisation condition for Kählers parameters $Q = \sqrt{\frac{q}{t}}\, t^N$ for which there is geometric transition:

GEOMETRIC ENGINEERING ON TORIC CY X



Using the refined topological vertex we can calculate the partition function of the Higgsed CY $\hat{X}$ and check that:

$$Z^{\hat{X}}_{\rm top}(\vec{\mu}, \vec{\tau}, q, t) = \mathcal{B}^{D_2 \times S^1, (\alpha_0)}_{FT[SU(N)]}(\vec{\mu}, \vec{\tau}, q, t)$$

$\vec{\mu}, \vec{\tau}$ are identified with fiber and base Kähler parameters.

3d duality from fiber-base duality

The CYs X, $\hat{X}$ (before and after Higgsing) are invariant under the action fiber-base duality which swaps μ_i with τ_i and so

$$
\begin{array}{ccc}
Z_{top}^{\hat{X}}(\vec{\mu}, \vec{\tau}, t) & \xleftrightarrow[\text{DUAL}]{\text{FIBER-BASE}} & Z_{top}^{\hat{X}}(\vec{\tau}, \vec{\mu}, t) \\
|| & & || \\
B_{FT[SU(N)]}^{D^2 \times S^1}(\vec{\mu}, \vec{\tau}, t) & \xleftrightarrow[\text{DUAL}]{\text{SPECTRAL}} & B_{FT[SU(N)]}^{D^2 \times S^1}(\vec{\tau}, \vec{\mu}, t)
\end{array}
$$

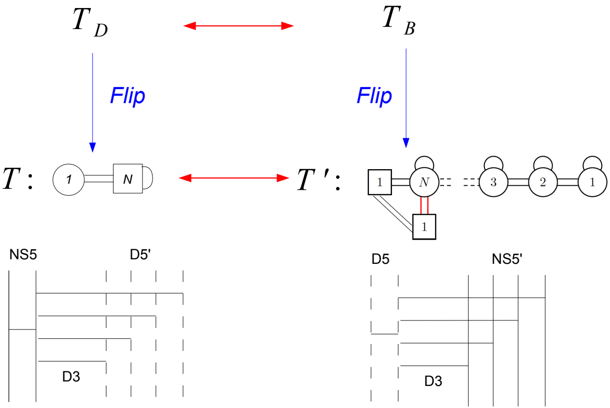
Notice that t is an Ω-background parameter not affected by the map.

$\rightarrow$ 3d self-duality for $FT[SU(N)]$ descends from fiber-base duality!

We can generate large families of new 3d dualities from fiber-base via Higgsing.

More spectral duals

Starting from the duality $\mathcal{T}_D \leftrightarrow \mathcal{T}_B$ we can obtain another spectral pair:



Viewing these theories as codimension-two defects also this spectral duality descends from 5d fiber base duality!

Gauge/q-CFT

HB of 3d $\mathcal{N}=2$ theories can be directly mapped to correlators of q-Toda vertex operators in the free boson Dotsenko-Fateev (DF) representation [Aganagic-Houzi-Shakirov].

For $FT[SU(N)]$ we have the following duality web (see Anton's talk)

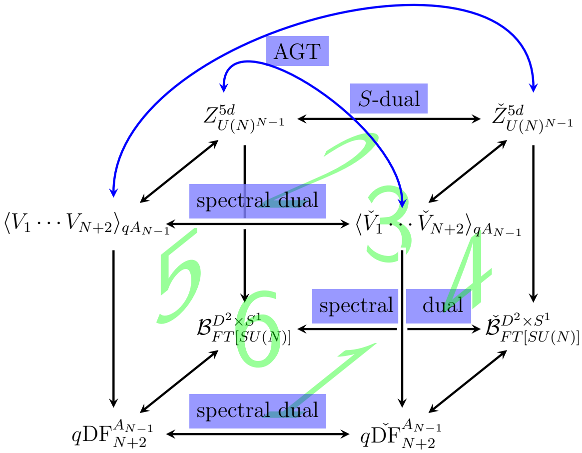
$$
\begin{array}{ccc}
\mathcal{B}^{D_2\times S^1}_{FT[SU(N)]} & \xleftrightarrow{\text{spectral dual}} & \check{\mathcal{B}}^{D_2\times S^1}_{FT[SU(N)]} \\
\text{gauge}/q\text{DF} \updownarrow & & \updownarrow \text{gauge}/q\text{DF} \\
q\text{DF}^{A_{N-1}}_{N+2} & \xleftrightarrow{\text{spectral dual}} & q\check{\text{DF}}^{A_{N-1}}_{N+2}
\end{array}
$$

- Horizontal arrows indicate (IR) dualities, requires highly non-trivial integral identities.
- Vertical arrows indicate AGT-like correspondenes. Trivial mapping, only need to establish a dictionary.

3d/q-CFT web via Higgsing

We saw that the $FT[SU(N)]$ spectral dual pair can be derived via Higgsing from 5d.

Similarly the $q\mathrm{DF}^{A_{N-1}}_{N+2}$ blocks can be obtained by tuning external and internal momenta in $\langle V_1 \cdots V_{N+2}\rangle_{q-A_{N-1}}$:



Now we focus again on our 3d/q-CFT web

$$
\begin{array}{ccc}
\mathcal{B}^{D_2\times S^1}_{FT[SU(N)]} & \xleftrightarrow{\text{spectral dual}} & \check{\mathcal{B}}^{D_2\times S^1}_{FT[SU(N)]} \\
\text{gauge}/q\text{DF}\ \updownarrow & & \updownarrow\ \text{gauge}/q\text{DF} \\
q\text{DF}^{A_{N-1}}_{N+2} & \xleftrightarrow{\text{spectral dual}} & q\check{\text{DF}}^{A_{N-1}}_{N+2}
\end{array}
$$

what happen when we take the $3d \to 2d$ or $q \to 1$?
See Anton's talk!

THANK YOU!

BACK-UP SLIDES

super-potentials

$$\mathcal{W}_{\mathcal{T}} = \hat{X}_{ij} q_i \tilde{q}_j$$

$$\mathcal{W}_{\mathcal{T}'} = (-)^N \left[d \underbrace{\Phi_N \cdots \Phi_N}_{N \text{ times}} \tilde{d} \right] \quad + u \Phi_N \tilde{u} + \mathcal{W}_{TSU(N)} +$$

$$+ S_+ u \tilde{d} + S_- d \tilde{u} + \sum_{m=2}^{N-1} \frac{\gamma_m}{m} Tr[d \phi^m \tilde{d}] \, .$$

singlets $S_\pm$ and $N-2$ γ_m and $Tr\mathcal{M}_{ij}$ are the motions $(2 \cdot 1 + 1 \cdot (N-1))$ in the brane set up.

RS Hamiltonians

Hamiltonians:

$$T_r(\vec{M}) = \sum_{\mathcal{I},|\mathcal{I}|=r} \prod_{i\in\mathcal{I},j\neq\mathcal{I}} \frac{\sinh \pi b(m_A - M_i - M_j)}{\sin \pi(M_i - M_j)} \prod_{j\in\mathcal{I}} e^{ib\partial_{M_j}} \,.$$

Eigenvalues:

$$\chi_r(\vec{T}) = \sum_{i_1<\cdots i_r} e^{2\pi b(T_{i_1}+\cdots T_{i_r})} \,.$$

Toda blocks

The A_n Toda theory describes $2d$ bosons $\vec{\phi}$ with $\sum_a e^{\sqrt{\beta}(\vec{\phi},\, \vec{e}_{(a)})}$ interaction.

Correlators of primary vertex operators $V_\alpha = e^{\frac{(\vec{\phi},\vec{\alpha})}{\sqrt{\beta}}}$ can be obtained in terms of free bosons correlators with insertion of N_a screening charges in each sector:

$$\langle \vec{\alpha}^{(\infty)} |\, V_{\vec{\alpha}^{(1)}}(z_1) \ldots V_{\vec{\alpha}^{(l)}}(z_l) \prod_{a=1}^{n} Q_{(a)}^{N_a} | \vec{\alpha}^{(0)} \rangle_{\text{free}} \,,$$

where

$$Q_{(a)} = \oint dx\, S_{(a)}(x) \,, \qquad\qquad S_{(a)} = e^{\sqrt{\beta}(\vec{\phi},\vec{e}_{(a)})} \,.$$

External and internal momenta must satisfy certain *quantization conditions.*

Dotsenko-Fateev integrals

After expanding in modes the free bosons

$$\phi^{(a)}(z) = Q^{(a)} + P^{(a)} \log z + \sum_{k\neq 0} c_k^{(a)} \frac{z^{-k}}{k}$$

with $[c_k^{(a)}, c_m^{(b)}] = k\delta_{k+m,0}\ \delta a, b$ and normal ordering, the correlators reduce to Dotsenko-Fateev integrals:

$$\mathrm{DF}_{l+2}^{A_n} \sim \oint \prod_{a=1}^{n} \prod_{i=1}^{N_a} dx_i^{(a)} \prod_{a=1}^{n} \prod_{i=1}^{N_a} \left(x_i^{(a)}\right)^{\beta(N_a - N_{a+1} - 1) + (\alpha_a^{(0)} - \alpha_{a+1}^{(0)})} \times$$

$$\prod_{a=1}^{n} \prod_{i\neq j}^{N_a} \left(1 - \frac{x_j^{(a)}}{x_i^{(a)}}\right)^{\beta} \prod_{a=1}^{n-1} \prod_{i=1}^{N_a} \prod_{j=1}^{N_{a+1}} \left(1 - \frac{x_j^{(a+1)}}{x_i^{(a)}}\right)^{-\beta} \prod_{p=1}^{l} \prod_{a=1}^{n} \prod_{i=1}^{N_a} \left(1 - \frac{x_i^{(a)}}{z_p}\right)^{\alpha_a^{(p)} - \alpha_{a+1}^{(p)}}$$

In the Virasoro (A_1) case these integrals can be calculated.
In the higher rank case integrals can be evaluated only for special values of the momenta.

q-deformed $\mathcal{W}_N$ algebras

Independently introduced by various groups in the 90s.

[Shiraishi-Kubo-Awata-Odake],[Lukyanov-Pugai],[Frenkel-Reshetikhin],[Jimbo-Miwa].

The $\mathcal{V}ir_{q,t}$ has generators T_n with $n \in \mathbb{Z}$, satisfying

$$[T_n\,,\ T_m] = -\sum_{l=1}^{+\infty} f_l\,(T_{n-l}\,T_{m+l} - T_{m-l}\,T_{n+l}) - \frac{(1-q)(1-t^{-1})(p^n-p^{-n})}{1-p}\delta_{m+n,0}$$

where $p = \frac{q}{t}$ and the functions f_l are determined by the expansion

$$f(z) = \sum_{l=0}^{+\infty} f_l z^l = \exp\left[\sum_{l=1}^{+\infty} \frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+p^n} z^n\right] = \frac{\left(qz;\frac{q^2}{t^2}\right)_\infty \left(t^{-1}z\frac{q^2}{t^2}\right)_\infty}{(1-z)\left(\frac{q^2}{t}z\frac{q^2}{t^2}\right)_\infty\left(\frac{q}{t^2}z;\frac{q^2}{t^2}\right)_\infty}\,.$$

The current $T(z) = \sum_n T_n z^{-n}$ satisfies

$$f\left(\frac{w}{z}\right)T(z)T(w) - f\left(\frac{z}{w}\right)T(w)T(z) = -\frac{(1-q)(1-t^{-1})}{1-\frac{q}{t}}\left(\delta\left(\frac{q}{t}\frac{w}{z}\right) - \delta\left(\frac{t}{q}\frac{w}{z}\right)\right)\,.$$

For $t = q^\beta$, $q = e^\hbar \to 1$ (β, z fixed) we recover the Virasoro current $L(z)$:

$$T(z) = 2 + \beta\hbar^2\left(z^2 L(z) + \frac{1}{4}\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)^2\right) + \ldots$$

q-Toda blocks

The q-deformation can be directly implemented on the free boson correlators and yields a q-deformed version of the Coulomb integrals:

$$q\mathrm{DF}^{A_n}_{l+2} \sim \oint \prod_{a=1}^{n}\prod_{i=1}^{N_a} dx_i^{(a)} \prod_{a=1}^{n}\prod_{i=1}^{N_a} \left(x_i^{(a)}\right)^{\beta(N_a-N_{a+1}-1)+(\alpha_a^{(0)}-\alpha_{a+1}^{(0)})+\sum_{p=1}^{l}\left(\alpha_a^{(p)}-\alpha_{a+1}^{(p)}\right)} \times$$

$$\times \prod_{a=1}^{n}\prod_{i\neq j}^{N_a} \frac{\left(\frac{x_j^{(a)}}{x_i^{(a)}};q\right)_\infty}{\left(t\frac{x_j^{(a)}}{x_i^{(a)}};q\right)_\infty} \prod_{a=1}^{n-1}\prod_{i=1}^{N_a}\prod_{j=1}^{N_{a+1}} \frac{\left(u\frac{x_j^{(a+1)}}{x_i^{(a)}};q\right)_\infty}{\left(v\frac{x_j^{(a+1)}}{x_i^{(a)}};q\right)_\infty} \prod_{p=1}^{l}\prod_{a=1}^{n}\prod_{i=1}^{N_a} \frac{\left(q^{1-\alpha_a^{(p)}}v^a\frac{z_p}{x_i^{(a)}};q\right)_\infty}{\left(q^{1-\alpha_{a+1}^{(p)}}v^a\frac{z_p}{x_i^{(a)}};q\right)_\infty},$$

where $u=\sqrt{qt}$ and $v=\sqrt{\frac{q}{t}}$.

This is exactly the $D^2\times S^1$ partition function of a quiver theory!

[Aganagic-Houzi-Shakirov]

$FT[SU(N)]$ block as a q-Toda block

The $FT[SU(N)]$ holomorphic block maps to an $N+2$-point A_{N-1} block

$$\mathcal{B}^{D_2\times S^1}_{FT[U(N)]} = q\mathrm{DF}^{A_{N-1}}_{N+2}\,,$$

with two generic primaries and $N-2$ are degenerates vertex operators $\vec{\alpha}^{(p)} = \beta\vec{\omega}_{N-1}$ for $p = 1, \cdots N-1$.

The dictionary is as follows:

$\mathcal{B}^{D_2\times S^1}_{FT[U(N)]}$	$q\mathrm{DF}^{A_{N-1}}_{N+2}$
Parameter $q = e^{\hbar} = q^{R\epsilon}$	Deformation parameter
Axial mass $m_A = t = q^{\beta}$	Central charge parameter β
Masses μ_p	Positions of the vertex operators z_p
FI parameters τ_a	Momentum vector $\vec{\alpha}^{(0)}_a$
Screening charges $N_a = a$	Ranks of the gauge groups

The spectral dual theory is mapped to the spectral dual q-block:

$$\mathcal{B}^{D_2\times S^1}_{\check{FT}[SU(N)]} = q\check{\mathrm{D}}\mathrm{F}^{A_{N-1}}_{N+2}\,.$$

Spectral duality on q-blocks exchanges insertions points with momenta.

$3d \to 2d$ reduction of the $FT[SU(N)]$ spectral dual pair

Recent thorough discussion in [Aharony-Razamat-Willett]:

- $2d$ limits depend on how the $3d$ real masses scale with R. We can obtain theories with $2d$ gauge theory or Landau-Ginsburg UV completion
- we can end up with direct sums of $2d$ theories
- earlier result: $3d$ abelian mirror pairs reduce to $2d$ Hori-Vafa dual pairs [Aganagic-Hori-Karch-Tong].
- we turn on all possible mass deformations so the $2d$ theories have isolated vacua (removing mass deformations is subtle if there are non-compact branches)

$FT[SU(N)]$ reduction

The *natural* Higgs limit reduces the $3d$ $FT[SU(N)]$ theory to the same theory in $2d$. The $3d$ FI parameters scale as $1/R$ and lift the Coulomb branch while the matter fields remain light.

In our conventions $q = e^{\hbar} \to 1$ ($\hbar = R\epsilon$) and

$$\tau_a = e^{RT_a} = \text{fixed}, \quad \mu_j = e^{RM_j} = q^{f_j}, \quad m_A = t = q^{\beta}\,.$$

The vacua remain at finite distances $x_i^{(a)} = e^{\hbar w_i^{(a)}} = q^{w_i^{(a)}}$.

Using

$$\lim_{q \to 1} \frac{(q^x;q)_\infty}{(q^y;q)_\infty} = (-\hbar)^{y-x}\,\frac{\Gamma(y)}{\Gamma(x)}\,,$$

the 1loop contributions of chiral and vector multiplets reduce to the D_2 contributions, the classical part also reduce and we find:

$$\lim_{q \to 1} \mathcal{B}^{D_2 \times S^1}_{FT[SU(N)]}(\vec{\tau}, \vec{\mu}, t) = \mathcal{B}^{D_2}_{FT[SU(N)]}(\vec{\tau}, \vec{f}, \beta)\,.$$

Dual $FT[SU(N)]$ reduction

On the dual side the Higgs limit becomes an *un-natural* Coulomb limit.

The vacua are at infinity so the $3d$ Coulomb brach parameters $x_i^{(a)}$ stay finite as $\hbar \to 0$.

Using

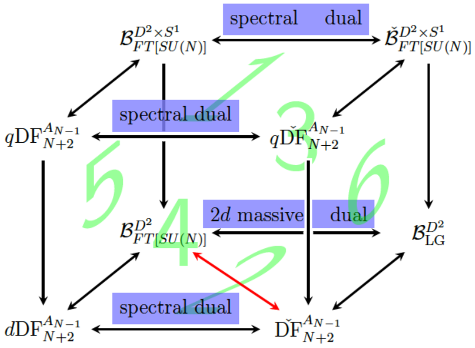
$$\lim_{q\to 1} \frac{(q^a x; q)_\infty}{(q^b x; q)_\infty} = (1-x)^{b-a} \,,$$

we immediately see that the dual block becomes a Selberg-like integral which can be immediately identified with a Toda DF block:

$$\lim_{q\to 1} \check{\mathcal{B}}^{D_2\times S^1}_{FT[SU(N)]}(\vec{\tau}, \vec{\mu}, t) = \check{DF}^{A_{N-1}}_{N+2} \,.$$

Gauge/CFT follows from 3d spectral duality!

Reduction of the 3d/q-CFT web



- $\mathcal{B}^{D_2}_{LG}$ is the D^2 partition function of a theory of twisted chiral multiplets with a twisted superpotential.
- $dDF^{A_{N-1}}_{N+2}$ is a correlator of vertex operators with d-$\mathcal{W}_N$ symmetry, an *un-natural* limit of q-$\mathcal{W}_N$
- The red link is the familiar gauge/CFT correspondence connecting surface operators vortex partition functions to Toda degenerate correlators.

d-Virasoro

Starting from the q-Virasoro relation:

$$f\left(\frac{w}{z}\right)T(z)T(w)-f\left(\frac{z}{w}\right)T(w)T(z)=-\frac{(1-q)(1-t^{-1})}{1-\frac{q}{t}}\left(\delta\left(\frac{q}{t}\frac{w}{z}\right)-\delta\left(\frac{t}{q}\frac{w}{z}\right)\right)$$

We can also take an *un-natural* limit where $z=q^u, w=q^v$ and finite $t(u)=\lim_{q\to 1}T(q^u)$ to obtain the d-Virasoro algebra:

$$g(v-u)t(u)t(v)-g(u-v)t(v)t(u)=\frac{\beta}{\beta-1}\left(\delta(v-u+1-\beta)-\delta(v-u-1+\beta)\right)$$

with

$$g(u)=\frac{2(1-\beta)}{u}\frac{\Gamma\left(\frac{u+2-\beta}{2(1-\beta)}\right)\Gamma\left(\frac{u+1-2\beta}{2(1-\beta)}\right)}{\Gamma\left(\frac{u+1}{2(1-\beta)}\right)\Gamma\left(\frac{u-\beta}{2(1-\beta)}\right)}.$$

We can find a bosonization of this algebra: find screening current and vertex operators and calculate their normal ordered correlators $dDF_{N+2}^{A_{N-1}}$.