Twisted Hilbert Space of 3d Supersymmetric Gauge Theories

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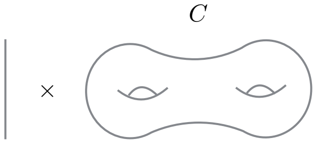
Introduction 1

I will talk about quantum field theories with 3d $\mathcal{N} = 2$ supersymmetry and an unbroken R-symmetry.



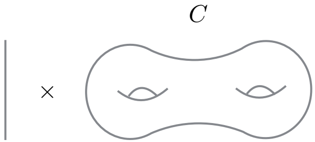
- Perform topological twist on C using R-symmetry.
- Supersymmetry algebra

$$\{Q, \bar{Q}\} = H - m_f \cdot J_f$$

  where H is hamiltonian and J_f is flavour charge.
- m_f is real mass parameter for flavour symmetry.

Introduction

I want to explain how to compute the ‘twisted Hilbert space’ $\mathcal{H}$ of supersymmetric ground states annihilated by Q, $\bar{Q}$.



- The supersymmetric ground states are graded by $(-1)^F$ and J_f.
- It should reproduce the twisted index on $S^1 \times C$,

$$I = \mathrm{Tr}_{\mathcal{H}} (-1)^F x^{J_f} \ .$$

- The latter can be computed by supersymmetric localisation.[1]

[1] Benini-Zaffaroni[2], Closset-Kim

Introduction

Why is the twisted Hilbert space is a richer observable?

1. There may be cancelations when computing the trace,

$$\bigoplus_{j=0}^{g} \wedge^{j}(\mathbb{C}^{g}) \quad \longrightarrow \quad \sum_{j=1}^{g}(-1)^{j}\binom{g}{j} = 0\,.$$

2. The supersymmetric ground states may exhibit 'wall-crossing'

$$\mathcal{H} = \begin{cases} \bigoplus\limits_{j=0}^{\infty} x^{j}\mathbb{C} & |x|<1 \\ -\bigoplus\limits_{j=0}^{\infty} x^{-j-1}\mathbb{C} & |x|>1 \end{cases} ,$$

whereas the supersymmetric index does not

$$\rightarrow \quad \begin{cases} \sum\limits_{j=0}^{\infty} x^{j} & |x|<1 \\ -\sum\limits_{j=0}^{\infty} x^{-j-1} & |x|>1 \end{cases} = \frac{1}{1-x} \quad .$$

Introduction

3. Turn on a superpotential $W(u)$ depending on complex parameters u.
   - The twisted index is independent of u.
   - The twisted Hilbert space is a holomorphic sheaf on the parameter space of u's.

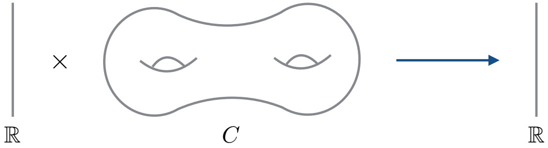
4. Turn on holomorphic line bundle for a $U(1)$ flavour symmetry on C.
   - The twisted index depends only on the flux

$$\frac{1}{2\pi}\int_C F = \mathfrak{m} \in \mathbb{Z}$$

   - The twisted Hilbert space is a holomorphic sheaf on the parameter space $\mathrm{Pic}^{\mathfrak{m}}(C)$.

Strategy

Introduce an 'effective' supersymmetric quantum mechanics that exactly captures supersymmetric ground states.



The supersymmetric quantum mechanics is of type $\mathcal{N} = (0, 2)$.

- Part 1: Supersymmetric Quantum Mechanics.
- Part 2: Three-dimensional $\mathcal{N} = 2$ Theories.

Part 1: Supersymmetric Quantum Mechanics

Supersymmetry Algebra

The supersymmetry algebra is

$$\{ Q , Q \} = 0$$
$$\{ Q , \bar{Q} \} = H - m_f \cdot J_f$$
$$\{ \bar{Q} , \bar{Q} \} = 0 .$$

- Flavour symmetry G_f with charge operator J_f.
- Real mass parameter $m_f \in \mathfrak{t}_f$.

Hilbert space of supersymmetric ground states $\mathcal{H}$,

$$Q \, |\psi\rangle = 0 \qquad \bar{Q} \, |\psi\rangle = 0 .$$

- Graded by fermion number $(-1)^F$ and flavour charge J_f.
- If spectrum is gapped, equivalent to cohomology of $\bar{Q}$.

Chiral Multiplet

Chiral multiplet (ϕ, ψ) with flavour symmetry $G_f = U(1)$ and associated mass parameter $m_f \in \mathbb{R}$.

$$Q = \psi \left(-\frac{\partial}{\partial \phi} + m_f \bar{\phi} \right) \qquad \bar{Q} = \bar{\psi} \left(+ \frac{\partial}{\partial \bar{\phi}} + m_f \phi \right) .$$

- The supercharges and $H - m_f J_f$ are unambiguous.
- There is a normal ordering ambiguity,

$$H \to H + \alpha m \qquad J_f \to J_f + \alpha \, .$$

- This is choice of background supersymmetric Chern-Simons term for $G_f = U(1)$ flavour symmetry.

Chiral Multiplet: Hilbert Space

The supersymmetric ground states are

$\mathfrak{c}_-$	$\mathfrak{c}_+$
$m_f < 0$	$m_f > 0$
$e^{m_f\|\phi\|^2}\bar{\phi}^j\bar{\psi}$	$e^{-m_f\|\phi\|^2}\phi^j$

- Introduce parameter $x \in \mathbb{C}^*$ to keep track of flavour charge.
- The supersymmetric ground states depends on the chamber,

$$\mathcal{H} = \begin{cases} x^{\alpha+\frac{1}{2}} \bigoplus_{j=0}^{\infty} x^j \mathbb{C} & m_f > 0 \\ -x^{\alpha-\frac{1}{2}} \bigoplus_{j=0}^{\infty} x^{-j} \mathbb{C} & m_f < 0 \end{cases} .$$

- Wall-crossing at $m_f = 0$ where spectrum not gapped.

Chiral Multiplet: Index

The supersymmetric index is

$$I = \begin{cases} x^{\alpha+\frac{1}{2}} \sum_{j=0}^{\infty} x^{j} & m_f > 0 \\ -x^{\alpha-\frac{1}{2}} \sum_{j=0}^{\infty} x^{-j} & m_f < 0 \end{cases} .$$

- Path integral construction identifies $x = e^{-2\pi\beta(m_f + iA_f)}$.
- These are expansions of the same rational function

$$\frac{x^{\alpha+\frac{1}{2}}}{1-x}$$

  in each chamber.
- Simple pole at $m_f = 0$ where spectrum not gapped.

Gauge Theory

Example [2]:

- $U(1)$ vectormultiplet $(A_\tau, \sigma, \lambda, \bar{\lambda}, D)$.
- N chiral multiplets $(\phi_1, \ldots, \phi_N)$ of charge $+1$.
- Real FI parameter $\zeta > 0$: $L_{\mathrm{FI}} = -\zeta D$.
- Supersymmetric Wilson line of charge q: $L_{\mathrm{WL}} = q(A_\tau + \sigma)$.

Global anomaly cancellation: $q - \frac{N}{2} \in \mathbb{Z}$. (I will assume $q - \frac{N}{2} \geq 0$.)

Flavour symmetry $G_f = PSU(N)$.

[2] Hori-Kim-Yi

Gauge Theory: Sigma Model Description

Classical potential: $$U = \sum_j |\sigma\phi_j|^2 + \frac{e^2}{2}(\sum_j |\phi_j|^2 - \zeta)^2 \quad .$$

Supersymmetric ground states captured by a sigma model to

$$M = \Big\{ \sum_{j=1}^{N} |\phi_j|^2 = \zeta \Big\} / U(1) = \mathbb{CP}^{N-1} \, .$$

- Supersymmetric Wilson line generates line bundle $\mathcal{O}(q)$.
- Quantization of fermions contributes $K_M^{1/2} = \mathcal{O}(-\frac{N}{2})$.
- Combination $F = \mathcal{O}(q - \frac{N}{2})$.

The wavefunctions are smooth sections of

$$\Omega^{0,*}(M) \otimes F \qquad \langle \alpha, \beta \rangle = \int \bar{\alpha} \wedge *\beta \, .$$

Gauge Theory: Hilbert Space

Turning on mass parameters $m_f = (m_1, \ldots, m_N)$,

$$Q = e^{h_f}\,\bar{\partial}^\dagger\, e^{-h_f} \qquad \bar{Q} = e^{-h_f}\,\bar{\partial}\, e^{h_f}$$

where $h_f = m_f \cdot \mu_f$ is moment map for infinitesimal $G_f = PSU(N)$ transformation generated by m_f.

- Spectrum is always gapped as target space compact.
- Setting $m_f = 0$ find symmetric tensor representation of G_f,

$$\begin{aligned}\mathcal{H} &= H^{0,\bullet}_{\bar{\partial}}(M, F) \\ &= S^{q-\frac{N}{2}}(x_1\mathbb{C} \oplus \cdots \oplus x_N\mathbb{C}) \,.\end{aligned}$$

- Supersymmetric index is character of this representation,

$$I = \chi_{S^{q-\frac{N}{2}}\mathbb{C}^N}(x_1, \ldots, x_N) \,.$$

Geometric Model

A massive supersymmetric sigma model specified by:

- A Kähler manifold M with isometry group G_f.
- A G_f-equivariant $\mathbb{Z}_2$-graded hermitian vector bundle F with odd differential $\delta : F \to F$.
- Real mass parameters $m_f \in \mathfrak{t}_f$.

The wavefunctions are smooth sections of

$$\Omega^{0,\bullet}(M) \otimes F$$

with hermitian inner product

$$\langle \alpha, \beta \rangle = \int_{\mathcal{M}} \bar{\alpha} \wedge * \beta \,.$$

Geometric Model

Supercharges are conjugated Dolbeault operators,

$$Q = e^{h_f}\, \bar\partial_F^\dagger\, e^{-h_f} + \delta^\dagger$$
$$\bar Q = e^{-h_f}\, \bar\partial_F\, e^{h_f} + \delta\,.$$

where $h = m_f \cdot \mu_f$ is moment map for infinitesimal G_f transformation generated by m_f.

- Supersymmetric ground states,

$$\mathcal{H} = H^{0,\bullet}_{\bar Q}(M, F)\,.$$

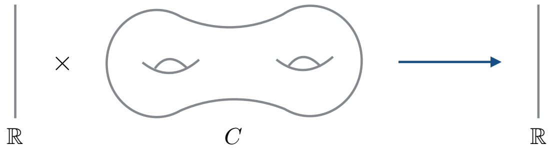
- If M is compact, spectrum is always gapped and $\mathcal{H}$ independent of $m_f \in \mathfrak{t}_f$.
- If M is non-compact, $\mathcal{H}$ exhibits 'wall-crossing' across loci where fixed point set of $m_f \in \mathfrak{t}_f$ is not-compact and spectrum not gapped.

Part 2: Supersymmetric Theories in Three Dimensions

3d $\mathcal{N} = 2$ Supersymmetry

Supersymmetry algebra,

$$\{Q_\alpha, \bar{Q}_\beta\} = P_{\alpha\beta} + (m_f \cdot J_f)\epsilon_{\alpha\beta} \,.$$



- Topological twist on C using $U(1)$ R-symmetry.
- Preserves supersymmetric quantum mechanics of type $\mathcal{N} = (0, 2)$,

$$\{Q, \bar{Q}\} = H - m_f \cdot J_f \,.$$

Chiral Multiplet

Chiral multiplet (ϕ, ψ_α, F) of R-charge r.

Decompose into supersymmetric quantum mechanics multiplets

- Chiral multiplet (ϕ, ψ) in section of $K_C^{r/2} \otimes L_f$
- Fermi multiplet (η, F) in $(0,1)$-form section of $K_C^{r/2} \otimes L_f$

where L_f is a holomorphic line bundle of degree $\mathfrak{m}_f$ on C associated to $U(1)_f$ flavour symmetry.

An E-term superpotential $E = \bar{D}\phi$ generates kinetic terms along C,

$$|E|^2 = \int_C \|\bar{D}\phi\|^2 \qquad \bar{\eta}\frac{\partial E}{\partial \phi}\psi = \int_C \bar{\eta} \wedge \bar{D}\psi \,.$$

Chiral Multiplet: Hilbert Space

Minimize classical potential: $\bar{D}\phi = 0$.

Fluctuations:

- Chiral multiplets in $H^0(K_C^{r/2} \otimes L_f)$: $\phi_1, \ldots, \phi_{n_C}$
- Fermi multiplets in $H^1(K_C^{r/2} \otimes L_f)$: $\eta_1, \ldots, \eta_{n_F}$
- Riemann-Roch: $n_C - n_F = (r-1)(g-1) + \mathfrak{m}_f$

Quantizing in the chamber $m_f > 0$, we find

$$\mathcal{H} = x^{\frac{n_C - n_F}{2}} \bigoplus_{j=0}^{\infty} x^j \bigoplus_{p+q=j} S^p(\mathbb{C}^{n_C}) \otimes \wedge^q(\mathbb{C}^{n_F}) \,.$$

- Depends on L_f through individual numbers n_C and n_F.
- (Can be promoted to sheaf of graded vector spaces on parameter space $\mathrm{Pic}^{\mathfrak{m}_f}(C)$ of L_f.)

Chiral Multiplet: Index

The twisted supersymmetric index is computed from trace,

$$
\begin{aligned}
I &= x^{\frac{n_C - n_F}{2}} \bigoplus_{j=0}^{\infty} x^j \binom{n_C - n_F + j - 1}{n_C - n_F} \\
&= \left(\frac{x^{1/2}}{1-x} \right)^{n_C - n_F} ,
\end{aligned}
$$

in agreement with 1-loop determinant from supersymmetric localisation.[3]

- Twisted supersymmetric index depends only on the difference

$$
n_C - n_F = (r-1)(g-1) + \mathfrak{m}_f \,.
$$

- It is therefore constant as L_f varies in parameter space $\mathrm{Pic}^{\mathfrak{m}_f}(C)$.

[3] [Benini-Zaffaroni[2],Closset-Kim]

Vectormultiplet

Three-dimensional vectormultiplet decomposes into the following 1d $\mathcal{N} = (0, 2)$ supermultiplets:

- A vectormultiplet for the group of gauge transformations $g : C \to U(1)$ with auxiliary field $D_{1d} = D + *_C F$.
- An adjoint chiral multiplet with complex scalar $\bar{D}_{\bar{z}}$.

In addition:

- A 3d FI parameter ζ contributes

$$-\zeta \int_C D = -\zeta \int_C D_{1d} + 2\pi\zeta\mathfrak{m}$$

- A 3d CS term contributes a supersymmetric Wilson line

$$\frac{k}{2\pi} \int_C (\sigma + iA_\tau) F$$

Example: $U(1)_{1/2}$ + 1 Chiral

Consider the following model:

- $U(1)$ supersymmetric Chern-Simons theory at level $+\frac{1}{2}$
- Chiral multiplet ϕ of charge $+1$ and R-charge $+1$

$U(1)_T$ topological flavour symmetry.

	$U(1)$	$U(1)_T$	$U(1)_R$
ϕ	$+1$	0	$+1$
T	0	$+1$	0

(This is mirror to single chiral multiplet - the monopole operator T.)

Important

This theory has only 'Higgs branch' vacua.

Sigma Model Description

The supersymmetric quantum mechanics has potential

$$
\begin{aligned}
U = &\int_C \|\bar{D}\phi\|^2 \\
&+ \int_C \|\frac{1}{e_{eff}^2} * F + |\phi|^2 - \frac{1}{2\pi}\xi_{eff}(\sigma)\|^2 \\
&+ \int_C \|\sigma\phi\|^2
\end{aligned}
$$

where

$$
\xi_{eff}(\sigma) = \begin{cases} \zeta + \sigma & \sigma > 0 \\ \zeta & \sigma < 0 \end{cases}.
$$

The potential is minimized by 'Higgs branch' vortices on C,

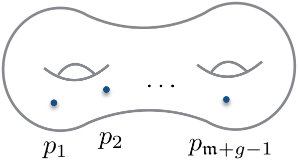
$$
\frac{1}{e_{eff}^2} * F + |\phi|^2 = \zeta \qquad \bar{D}\phi = 0 \qquad \sigma = 0\,.
$$

Moduli Space

Let $\mathcal{M}_{\mathfrak{m}}$ denote the moduli space of solutions with flux $\mathfrak{m}$.

This has an algebraic description:

- A holomorphic line bundle L of degree $\mathfrak{m}$.
- A non-vanishing section $\phi \in H^0(K_C^{1/2} \otimes L)$.



We can parametrise moduli space by effective divisor of ϕ:

$$\mathcal{M}_{\mathfrak{m}} = \begin{cases} \mathrm{Sym}^{\mathfrak{m}+g-1} C & \text{if} \quad \mathfrak{m} \geq 1-g \\ \emptyset & \text{if} \quad \mathfrak{m} < 1-g \,. \end{cases}$$

Hilbert Space

Supersymmetric ground states captured by a supersymmetric quantum mechanics to $\mathcal{M}_{\mathfrak{m}}$,

$$\mathcal{H} = \bigoplus_{\mathfrak{m} \geq 1-g} H^{0,\bullet}_{\bar{\partial}}(\mathcal{M}_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}})\,.$$

The holomorphic line bundle $\mathcal{F}$ receives contributions from:

- Fermion fluctuations: $K^{1/2}_{\mathcal{M}_{\mathfrak{m}}}$.
- Supersymmetric Chern-Simons term: $K^{-1/2}_{\mathcal{M}_{\mathfrak{m}}}$.
- A holomorphic line bundle L_T on C for the flavour topological symmetry is an 'electric impurity': it induces a line bundle $\widetilde{L}_T$ on the vortex moduli space $\mathcal{M}_{\mathfrak{m}}$.

Hilbert Space

In the absence of L_T, the space of supersymmetric vacua is

$$\mathcal{H} = \bigoplus_{\mathfrak{m} \geq 1-g} x^{\mathfrak{m}} \, H^{0,\bullet}_{\bar{\partial}}(\mathcal{M}_{\mathfrak{m}})$$
$$= \bigoplus_{\mathfrak{m} \geq 1-g} x^{\mathfrak{m}} \bigoplus_{j=0}^{\mathfrak{m}+g-1} \wedge^j (\mathbb{C}^g) \,.$$

- The cohomology of a symmetric product is an exterior algebra

$$H^{0,j}(\mathcal{M}_{\mathfrak{m}}) = \wedge^j (\mathbb{C}^g) \,.$$

- The generators are inherited from the curve, $H^{0,1}_{\bar{\partial}}(C) = \mathbb{C}^g$.
- There an infinite number of supersymmetric ground states!

Index

The twisted supersymmetric index truncates to a finite Laurent polynomial,

$$I \;=\; \sum_{\mathfrak{m}\geq 1-g} x^{\mathfrak{m}} \sum_{q=0}^{\mathfrak{m}+g-1} (-1)^q \binom{g}{q} \;=\; x^{1-g}(1-x)^{g-1}\,.$$

- Supersymmetric ground states with $\mathfrak{m} > 0$ cancel out.
- This coincides with the contour integral from supersymmetric localization [4],

$$I = \sum_{\mathfrak{m}\in\mathbb{Z}} (-x)^{\mathfrak{m}} \int_{\Gamma} \frac{dz}{z} \frac{z^{\mathfrak{m}}}{(1-z)^{\mathfrak{m}+g}}\,.$$

- Localisation formula reinterpreted as Hirzebruch-Riemann-Roch for holomorphic Euler character, $\chi(\mathcal{M}_{\mathfrak{m}})$.

[4] Benini-Zaffaroni

Mirror Symmetry

Consider the following mirror pair:

- $U(1)_{1/2}$ + 1 Chiral.
- 1 Chiral + mixed supersymmetric Chern-Simons terms $k_{ff} = k_{Rf} = -\frac{1}{2}$.

The supersymmetric ground states match,

$$\mathcal{H} = x^{1-g} \bigoplus_{j \geq 0} x^j \bigoplus_{q=0}^{j} \wedge^q (\mathbb{C}^g) \, .$$

- This is a stronger check than the supersymmetric the twisted index!
- (Introducing a line bundle for $U(1)$ flavour symmetry, agreement of sheaves of graded vector spaces on parameter space $\mathrm{Pic}(C)$.)

General Structure

For a $U(1)$ supersymmetric gauge theory with only 'Higgs branch' vacua,

$$\mathcal{H} = \bigoplus_{\mathfrak{m} \in \mathbb{Z}} H^{0,\bullet}_{\bar{Q}}(\mathcal{M}_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}})$$

- $\mathcal{M}_{\mathfrak{m}}$ = moduli space of vortex equations on C with flux $\mathfrak{m}$.
- $\mathcal{F}_{\mathfrak{m}} = \mathbb{Z}_2$-graded G_f-equivariant vector bundle with contributions from fermions, Chern-Simons terms and line bundles for topological symmetries.
- $\delta_{\mathfrak{m}} : \mathcal{F}_{\mathfrak{m}} \to \mathcal{F}_{\mathfrak{m}}$ is odd differential from 3d superpotential.
- The supercharge is

  $$\bar{Q} = e^{-h_f} \bar{\partial}_{\mathcal{F}_{\mathfrak{m}}} e^{h_f} + \delta_{\mathfrak{m}}$$

  where $h = m_f \cdot \mu_f$ is the moment map for the infinitesimal G_f transformation generated by mass parameters m_f.

Future Directions

- Theories with 'topological vacua' require further analysis!
- Enumeration and action of local operators in supersymmetric quantum mechanics.
- Inclusion of supersymmetric line operators.
- States defined by boundary conditions / interfaces.
- $\mathcal{N} = 4$ theories and connections to conformal blocks for vertex operator algebras? [5]
- Action of $SL(2, \mathbb{Z})$ on twisted Hilbert spaces of theories with $U(1)$ flavour symmetry? [6]
- A homological version of the 3d-3d correspondence? [7]

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[5] Gaiotto, Gaiotto-Kostello

[6] Witten

[7] Gukov-Putrov-Vafa

Thank you for listening!

Questions?