Wilson loops in 5d $\mathcal{N} = 1$ theories and S-duality

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with Antonio Sciarappa (KIAS).

- SCFTs in 5 dimensions have no marginal deformations, but they have massive deformations. The mass deformed theories can (sometimes) be described as SYM gauge theories with massive (real) parameters,

- gauge couplings $t = \frac{1}{g_{\mathrm{YM}}^2}$;
- hypermultiplet masses m;
- adjoint scalar vevs a.

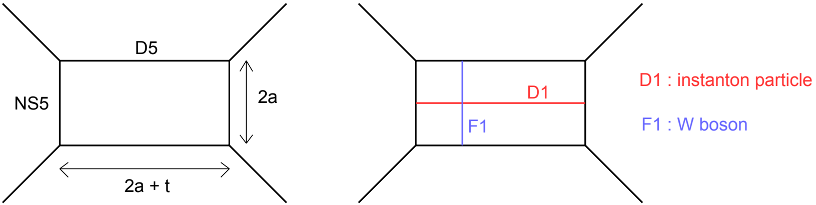
- The SCFT point in parameter space is at $t = m = a = 0$ (infinite coupling).

- It happens that two deformations of a single SCFT are described by two different gauge theories, leading to a notion of duality.

- One such duality is implemented by S-duality in the IIB brane realization of 5d $\mathcal{N}=1$ gauge theories. It relates $SU(N)^{M-1}$ quiver theories to $SU(M)^{N-1}$ quiver theories.

- The simplest case: $SU(2)_{t,a} \leftrightarrow SU(2)_{\tilde{t},\tilde{a}}$ self-duality with

$$\tilde{t} = -t\,, \quad \tilde{a} = a + t/2\,.$$

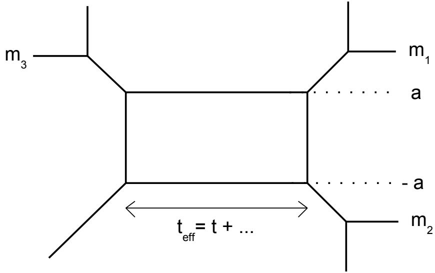


- S-duality = diagonal reflection in the figure. It relates positive to negative coupling! But observables should make sense for $a > 0$ and $t > -2a$, allowing negative t (some overlap in parameter space).

We can add flavors. Seiberg '96

$SU(2)$ theory with $N_f \leqslant 7$ fundamental hypers → self-dual again.

Example: $N_f = 3$.



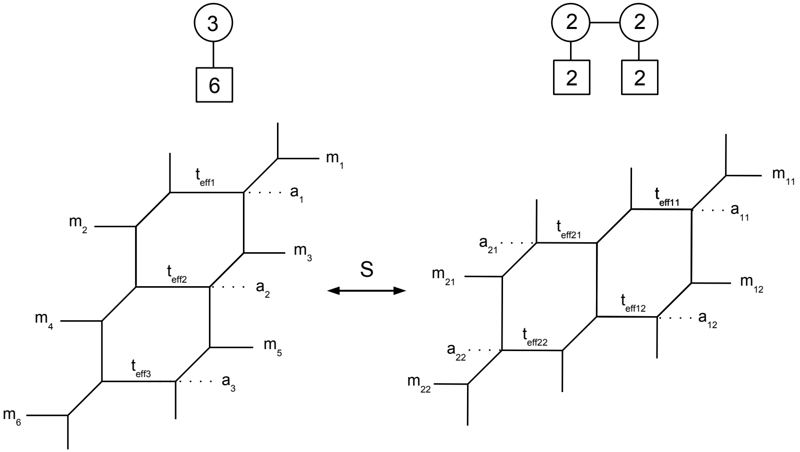
S-duality map

$$\tilde{a} = a + \frac{t}{2} - \frac{1}{2}(m_1 - m_2 + m_3)\,, \quad \tilde{t} = -\frac{t}{4} + \frac{5}{8}(m_1 - m_2 + m_3)\,,$$
$$\tilde{m}_1 = \frac{t}{2} + \frac{1}{4}(3m_1 + m_2 - m_3)\,, \quad \tilde{m}_2 = -\frac{t}{2} + \frac{1}{4}(m_1 - m_2 - 3m_3)\,,$$
$$\tilde{m}_3 = \frac{t}{2} - \frac{1}{4}(m_1 + 3m_2 + m_3)\,.$$

Symmetry enhancement $SO(2N_f) \times U(1)_{\rm inst} \to E_{N_f+1}$.

Kim et al '12, Bergman et al '13, Hwang et al '14, Ganor et al '16

Higher rank S-dual theories:



Test with SCI Bergman et al '13 and topological strings Mitev et al '14

Wilson loops

The gauge theories have half-BPS Wilson loop operators

$$W_{\mathcal{R}} = \mathrm{Tr}_{\mathcal{R}}\, \mathcal{P} \exp\Big(\int iA + \sigma dx \Big) \,,$$

They are labeled by a representation $\mathcal{R}$ of the gauge group.

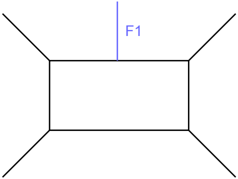
- A natural question: what are the S-dual loop operators? This is a priori not obvious because dual theories have different gauge groups (except for self-dual cases).

In 4d $\mathcal{N} = 4$ SYM: Wilson loops $\leftrightarrow$ 't Hooft loops under S-duality.
In 3d $\mathcal{N} = 4$ theories: Wilson loops $\leftrightarrow$ Vortex loops under mirror symmetry.

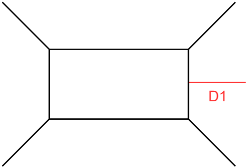
- To understand the duality action we try to realize the Wilson loops in the brane picture with extra strings/brane elements and we act with S-duality.

Half-BPS line operators can be realized by adding F1, D1 and/or D3.

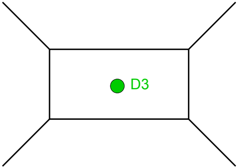
	0	1	2	3	4	5	6	7	8	9
D5	X	X	X	X	X	X				
NS5	X	X	X	X	X		X			
$5_{(p,q)}$	X	X	X	X	X	o	o			
F1	X						X			
D1	X					X				
D3	X							X	X	X



Wilson loop

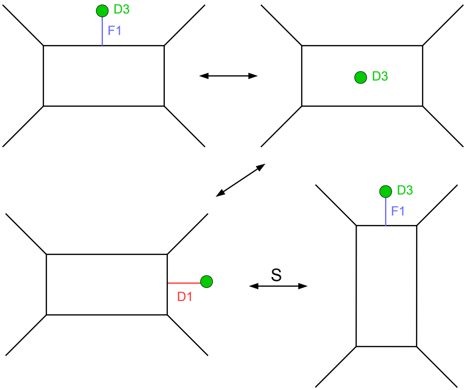


instanton loop



SQM loop

Acting with S-duality changes the Wilson loop (F1) into an instanton loop (D1). However, playing with Hanany-Witten moves, one understands that there are relations between these loop operators.



$\Rightarrow$ the fundamental Wilson loop is mapped to itself under S-duality!
This is also true for Wilson loops in $\mathcal{R} = \mathbf{2}^{\otimes n}$ (n F1s attached to n D3s).

We would like to test the S-duality conjecture $W_{\mathbf{2}^{\otimes n}} \leftrightarrow \widetilde{W}_{\mathbf{2}^{\otimes n}}$ with exact results. $\rightarrow$ $S^1 \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$ 'half-index' with loop on S^1.

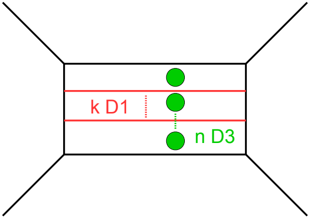
It has the form of an expansion in instanton charge, with coefficients given by ADHM QM partition functions.

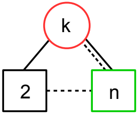
The direct computation is not fully understood (the loop affects the moduli space of instantons at the origin).

Instead we could compute certain SQM loops realized with D3 branes and identify the Wilson loops as special residues in flavor fugacities.
H-C Kim '16, Agarwal et al '18

The advantage of the SQM loops is that their brane realizations provide the modified ADHM data.

Brane configuration for k-instanton ADHM $\mathcal{N} = (0, 4)$ QM with n D3s
SQM loop ($2n$ 1d fermions):





Residue formula:

$$\langle W_{\mathbf{2}^{\otimes n}} \rangle = (-1)^n \oint_{\mathcal{C}} \prod_{i=1}^{n} \frac{dx_i}{2\pi i x_i} \langle L^n_{\text{SQM}} \rangle (x_1, \ldots, x_n) \, .$$

with $Q_F = e^{-2a}$, $Q_B = e^{-t-2a}$, we find

$$
\begin{aligned}
Q_F^{1/2}\langle W_{\mathbf{2}}\rangle &= 1 + Q_F + Q_B + \chi_{\mathbf{3}}^{A_1}(q_+)Q_F Q_B + \chi_{\mathbf{5}}^{A_1}(q_+)Q_F Q_B(Q_F + Q_B) \\
&+ Q_F Q_B(Q_F^2 + Q_F Q_B + Q_B^2)\chi_{\mathbf{7}}^{A_1}(q_+) \\
&+ Q_F^2 Q_B^2\Big(\chi_{\mathbf{7}}^{A_1}(q_+) + \chi_{\mathbf{5}}^{A_1}(q_+) + \chi_{\mathbf{2}}^{A_1}(q_-)\chi_{\mathbf{8}}^{A_1}(q_+)\Big) + \ldots,
\end{aligned}
$$

$$
\begin{aligned}
Q_F\langle W_{\mathbf{2}\otimes\mathbf{2}}\rangle &= 1 + 2(Q_F + Q_B) + \Big(Q_F^2 + Q_F Q_B + Q_B^2\Big) + \Big(\chi_{\mathbf{3}}^{A_1}(q_+) + \chi_{\mathbf{2}}^{A_1}(q_+)\chi_{\mathbf{2}}^{A_1}(q_-)\Big)Q_F Q_B \\
&+ Q_F Q_B(Q_F + Q_B)\Big(\chi_{\mathbf{5}}^{A_1}(q_+) + \chi_{\mathbf{3}}^{A_1}(q_+) + \chi_{\mathbf{4}}^{A_1}(q_+)\chi_{\mathbf{2}}^{A_1}(q_-)\Big) \\
&+ Q_F Q_B(Q_F^2 + Q_F Q_B + Q_B^2)\Big(\chi_{\mathbf{7}}^{A_1}(q_+) + \chi_{\mathbf{5}}^{A_1}(q_+) + \chi_{\mathbf{6}}^{A_1}(q_+)\chi_{\mathbf{2}}^{A_1}(q_-)\Big) \\
&+ Q_F^2 Q_B^2\Big(\chi_{\mathbf{8}}^{A_1}(q_+)\chi_{\mathbf{2}}^{A_1}(q_-) + \chi_{\mathbf{6}}^{A_1}(q_+)\chi_{\mathbf{2}}^{A_1}(q_-) + \chi_{\mathbf{4}}^{A_1}(q_+)\chi_{\mathbf{2}}^{A_1}(q_-) \\
&\qquad + \chi_{\mathbf{7}}^{A_1}(q_+)\chi_{\mathbf{3}}^{A_1}(q_-) + \chi_{\mathbf{7}}^{A_1}(q_+) + 2\chi_{\mathbf{5}}^{A_1}(q_+) + 1\Big) + \ldots,
\end{aligned}
$$

$$
\begin{aligned}
Q_F^{3/2}\langle W_{\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}}\rangle &= 1 + 3(Q_F + Q_B) + 3(Q_F^2 + Q_F Q_B + Q_B^2) \\
&+ \Big(\chi_{\mathbf{3}}^{A_1}(q_+) + \chi_{\mathbf{3}}^{A_1}(q_-) + \chi_{\mathbf{2}}^{A_1}(q_+)\chi_{\mathbf{2}}^{A_1}(q_-) + 2\Big)Q_F Q_B \\
&+ \Big(\chi_{\mathbf{5}}^{A_1}(q_+) + 3\chi_{\mathbf{3}}^{A_1}(q_+) + \chi_{\mathbf{3}}^{A_1}(q_+)\chi_{\mathbf{3}}^{A_1}(q_-) + \chi_{\mathbf{4}}^{A_1}(q_+)\chi_{\mathbf{2}}^{A_1}(q_-) \\
&\qquad + \chi_{\mathbf{2}}^{A_1}(q_+)\chi_{\mathbf{2}}^{A_1}(q_-)\Big)Q_F Q_B(Q_F + Q_B) + \Big(Q_F^3 + Q_F^2 Q_B + Q_F Q_B^2 + Q_B^3\Big) + \ldots.
\end{aligned}
$$

S-duality map : $Q_F \leftrightarrow Q_B$.

We find

$$\langle W_{\mathbf{2}^{\otimes n}} \rangle (Q_F, Q_B) = \left(\frac{Q_B}{Q_F} \right)^{n/2} \langle W_{\mathbf{2}^{\otimes n}} \rangle (Q_B, Q_F) \, .$$

The Wilson loops in representations $\mathbf{2}^{\otimes n}$ transform covariantly!

$$S.W_{\mathbf{2}^{\otimes n}} = e^{-\frac{nt}{2}} \, \widetilde{W}_{\mathbf{2}^{\otimes n}} \, .$$

The coefficient $e^{-\frac{nt}{2}}$ is a charge $-n$ background Wilson loop of $U(1)_{\text{inst}}$.

$\Rightarrow$ Wilson loops in other representations are mapped to combinations of Wilson loops (it can be worked out if necessary).

$\Rightarrow$ $\{W_{\mathbf{2}^{\otimes n}}, n \in \mathbb{N}\}$ is the natural basis to express the S-duality map.

We find qualitatively identical results for $SU(2)$ theories with $N_f \leqslant 4$ flavors.

$SU(3) \leftrightarrow SU(2)^2$ dualities

Similar considerations lead to the map of representations (natural basis for the duality map)

$$\mathbf{3}^{\otimes n_1} \otimes \overline{\mathbf{3}}^{\otimes n_2} \quad \leftrightarrow \quad (\mathbf{2}^{\otimes n_1}, \mathbf{2}^{\otimes n_2})\,,$$

and the exact computations reveal the relations

$$S.W_{\mathbf{3}^{\otimes n_1} \otimes \overline{\mathbf{3}}^{\otimes n_2}} = Y_1^{-n_1} Y_2^{-n_2} \widetilde{W}_{(\mathbf{2}^{\otimes n_1}, \mathbf{2}^{\otimes n_2})}\,,$$

with Y_1, Y_2 background Wilson loops of the global symmetry.

Explicitly checked for $SU(3)$, $N_f = 2, 4$, and $n_1 + n_2 \leqslant 2$ only.

Concluding comments

- The story generalizes to $SU(N)^{M-1} \leftrightarrow SU(M)^{N-1}$ dualities without conceptual difficulty.
- It is technically hard to go far in the computations.
- We have also checked the E_{N_f+1} symmetry enhancement in $SU(2)$ theories.
- We should explore the Wilson loop - instanton loop duality.