The Importance of Being Disconnected: Principal Extension Gauge Theories

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Work with Alessandro Pini and Diego Rodriguez-Gomez (1804.01108)
Gauge theories initially formulated using simple Lie algebras. Possible extensions include:

- Products of many Lie algebras / groups

\[ \text{SU}(3) \rightarrow \text{SU}(3) \rightarrow \text{SU}(3) \]

\[ [\text{Gaiotto}, \ 0904.2715] \]
Gauge theories initially formulated using simple Lie algebras. Possible extensions include:

- Products of many Lie algebras / groups

\[ \text{SU}(3) \to \text{SU}(3) \to \text{SU}(3) \]

- Global structure (\( \pi_1 \)) of the group

\[ \text{SU}(2) \quad \text{SO}(3)_+ \quad \text{SO}(3)_- \]

![Figure 1: The weights of line operators of gauge theories with \( g = \text{su}(2) \).](Gaiotto, 0904.2715)

- What about the global structure \( \pi_0 \) of the group?
Recent interest on discrete gauging (recall Hanany’s talk). Gauging of a discrete symmetry allows for new Coulomb branch geometries.

[Argyres, Martone, 1611.08602]
Introduction

Alternative approach: start from a disconnected gauge group.

In this work

- We consider a special class of non-connected groups
- We focus on 4d $\mathcal{N} = 2$ SCFTs
- We look at local physics and use algebraic counting tools.
Principal Extension Groups

The Coulomb branch Index

The Higgs Branch of SQCD and the Global Symmetry
Principal Extension Groups

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\[ \tilde{SU}(N) = SU(N) \ltimes \varphi \{1, \mathcal{P}\} \]

\[ \tilde{SO}(2N) = SO(2N) \ltimes \varphi \{1, \mathcal{P}\} = O(2N) \]
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\((X, 1)\)  \hspace{1cm}  \((X, \mathcal{P})\)

In the case of \(SU(N)\), this is related to complex conjugation:

\[ X = A^{-1} P (X) A, \quad A^T = (-1)^{N-1} A \]

and \(\det A = 1\).
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\[ \bar{X} = A^{-1} \mathcal{P}(X) A, \quad A^T = (-1)^{N-1} A \quad \text{and} \quad \det A = 1. \]
Representations ($\tilde{SU}(3)$ example)
Outline

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The Coulomb branch Index

The Higgs Branch of SQCD and the Global Symmetry
Recall the superconformal index for a theory with gauge group $G$ and some fundamental matter multiplets:

$$I = \int d\eta_G(X) \text{PE} \left[ \sum_{i \in \text{multiplets}} f^{R_i} \chi^{R_i}(X) \right];$$

where

$$f^{V} = -\frac{\sigma \tau}{1 - \sigma \tau} - \frac{\rho \tau}{1 - \rho \tau} + \frac{\sigma \rho - \tau^2}{(1 - \rho \tau)(1 - \sigma \tau)},$$

$$f^{1/2H} = \frac{\tau (1 - \rho \sigma)}{(1 - \rho \tau)(1 - \sigma \tau)}.$$
In the limit

$$\tau \to 0, \quad \rho \sigma =: t.$$ 

we have

$$f^V = t, \quad f^{\frac{1}{2}H} = 0.$$ 

This gives the **Coulomb branch index** [Gadde, Rastelli, Razamat, 1110.3740].
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This gives the \textit{Coulomb branch index} \cite{Gadde:2011ik}.

\[ \mathcal{I}_G^{\text{Coulomb}}(t) = \int_G d\eta_G(X) \frac{1}{\det (1 - t\Phi_{\text{Adj}}(X))}, \]
This is Molien’s formula for the Hilbert series of invariants of the adjoint representation.
What is a Hilbert Series

Let $R$ be a Noetherian graded ring with $R_0 = \mathbb{C}$,

$$R = \bigoplus_{n \in \mathbb{N}} R_n.$$

The Hilbert series of $R$ is

$$HS(R, t) = \sum_{n \in \mathbb{N}} t^n \dim_{\mathbb{C}} R_n.$$

Example:

$$HS(\mathbb{C}[x], t) = \frac{1}{1 - t}$$
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For polynomial rings, the Hilbert series is always a rational function [Hilbert, 1890.xxxx]
What is a Hilbert Series

If the ring is a *complete intersection*,

\[
\text{HS}(\mathbb{C}[\text{Gens}]/(\text{Rels}), t) = \prod_{\text{Rels}} \left(1 - t^{\deg(\text{Rels})}\right) \prod_{\text{Gens}} \left(1 - t^{\deg(\text{Gens})}\right)
\]
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$$

Example:

$$
\text{HS}\left(\mathbb{C}[x_1, x_2, x_3] / (x_1 x_2 - x_3^2), t\right) = 1 + 3t + 5t^2 + \ldots = \frac{1 - t^2}{(1 - t)^3}
$$
Basic Invariant Theory

Particular case: freely-generated ring

\[ \mathbb{C}[x]^G \cong \mathbb{C}[I_1, \ldots, I_m], \quad HS(\mathbb{C}[x]^G, t) = \frac{1}{m \prod_{i=1}^{m} (1 - t^{\deg I_i})}. \]
Particular case: freely-generated ring

$$\mathbb{C}[x]^G \cong \mathbb{C}[l_1, \ldots, l_m], \quad HS(\mathbb{C}[x]^G, t) = \frac{1}{m \prod_{i=1}^m (1 - t^{\deg l_i})}.$$ 

In general, Hironaka decomposition for invariant rings:

$$\mathbb{C}[x]^G \cong \bigoplus_{j=1}^p J_j \mathbb{C}[l_1, \ldots, l_m]$$

$$HS(\mathbb{C}[x]^G, t) = \frac{\sum_{j=1}^p t^{\deg J_j}}{m \prod_{i=1}^m (1 - t^{\deg l_i})}.$$
Consider a finite group $G$, and a (finite-dimensional complex) representation $V$. Consider the ring of invariants $\mathbb{C}[V]^G$. 
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Molien’s formula:

$$\text{HS} \left( \mathbb{C}[V]^G, t \right) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det_V(1 - t \cdot g)}.$$
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For a large class of groups ("reductive"),

$$\frac{1}{|G|} \sum_{g \in G} \longrightarrow \int_G d\eta_G(X).$$
For a class function $f$,
\[
\int_{\tilde{SU}(N)} d\eta_{\tilde{SU}(N)}(X)f(X) = \frac{1}{2} \left[ \int d\mu_+^N(z)f(z) + \int d\mu_-^N(z)f(z^P) \right]
\]

[Wendt, 1999]

with
\[
d\mu_+^N(z) = \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j} \prod_{\alpha \in \mathbb{R}^+(A_{N-1})} (1 - z(\alpha)) ,
\]

and
\[
N \text{ even: } d\mu_-^N(z) = \prod_{j=1}^{N/2} \frac{dz_j}{2\pi i z_j} \prod_{\alpha \in \mathbb{R}^+(B_{N/2})} (1 - z(\alpha)) .
\]

\[
N \text{ odd: } d\mu_-^N(z) = \prod_{j=1}^{(N-1)/2} \frac{dz_j}{2\pi i z_j} \prod_{\alpha \in \mathbb{R}^+(C_{(N-1)/2})} (1 - z(\alpha)) .
\]
The Coulomb Index

Computation for $SU(N)$:

$$\mathcal{I}_{SU(N)}^{\text{Coulomb}}(t) = \frac{1}{\prod_{i=2}^{N}(1 - t^i)},$$

corresponds to

$$\mathbb{C}[\phi_{ij}]^{SU(N)} \cong \mathbb{C}[^{\text{Tr}}(\phi^k)_{k=2,…,N}],$$

polynomial ring without any relation.
Computation for $\widetilde{SU}(N)$:

$$\mathcal{I}_{\text{Coulomb}}^{\text{SU}(N)}(t) = \frac{\sum_{k_1 < \cdots < k_r \text{ odd}} t^{k_1 + \cdots + k_r}}{\prod_{i \text{ even}} (1 - t^i) \prod_{i \text{ odd}} (1 - t^{2i})},$$

Why?

$$\text{Tr}((\phi^P)^k) = (-1)^k \text{Tr}(\phi^k).$$

There are "holes" in the structure of invariants.
Invariant theory interpretation:

1. The primary invariants $I_k$ for $2 \leq k \leq N$ defined by

   
   \[
   I_k = \begin{cases} 
   \text{Tr}(\phi^k) & \text{for } k \text{ even} \\
   \text{Tr}(\phi^k)^2 & \text{for } k \text{ odd}
   \end{cases}
   \]

2. The secondary invariants $J_{k_1, \ldots, k_r}$ for $k_1, \ldots, k_r$ odd and $3 \leq k_1 < \cdots < k_r \leq N$, with $r$ even ($r = 0$ corresponds to the trivial invariant 1).

   \[
   J_{k_1, \ldots, k_r} = \prod_{i=1}^{r} \text{Tr}(\phi^{k_i})
   \]

   Relations (among others):

   \[
   J_{k_1, \ldots, k_r}^2 - I_{k_1} \cdots I_{k_r} = 0,
   \]
For $N \geq 5$ the Coulomb branch of $\mathcal{N} = 2 \tilde{\mathbf{SU}}(N)$ gauge theories (of genus 0) is not freely generated.

See also [Argyres, Martone, 1804.03152]
[Bourton, Pini, Pomoni, 1804.05396]
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Example for $N = 5$: Coulomb branch parametrized by $I_2, I_3, I_4, I_5, J_{3,5}$ with relation $J_{3,5}^2 = I_3 I_5$. 
For $N \geq 5$ the Coulomb branch of $\mathcal{N} = 2$ $\widetilde{SU}(N)$ gauge theories (of genus 0) is not freely generated.

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Example for $N = 5$: Coulomb branch parametrized by $I_2, I_3, I_4, I_5, J_{3,5}$ with relation $J_{3,5}^2 = I_3 I_5$.

Complex singularity of complex dimension 2 parametrized by $I_2$ and $I_4$. 
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The Higgs branch is

$$\left( \mathbb{C}[Q, \bar{Q}]/(\mathbb{C}F\text{-terms}) \right)^{\text{Gauge group}}.$$
Higgs branch Hilbert Series

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Consider the \( \mathcal{N} = 2 \) \( SU(N) \) gauge theory with \( 2N \) fundamental hypers.

\[
\mathcal{W} \sim \text{Tr} \bar{Q} \phi Q \implies Q\bar{Q} - \frac{1}{N} (\text{Tr} Q\bar{Q}) 1_N = 0
\]
The Higgs branch is

\[ \left( \mathbb{C}[Q, \tilde{Q}] / (\mathbb{C}F\text{-terms}) \right)^{\text{Gauge group}}. \]

Consider the \( \mathcal{N} = 2 \ SU(N) \) gauge theory with \( 2N \) fundamental hypers.

\[ \mathcal{W} \sim \text{Tr} \tilde{Q} \phi Q \quad \implies \quad Q \tilde{Q} - \frac{1}{N} (\text{Tr} Q \tilde{Q}) 1_N = 0 \]

Higgs branch Hilbert series:

\[ H_{SU(N)} = \int d\eta_{SU(N)}(X) \frac{\det (1 - t^2 \Phi_{\text{Adj}}(X))}{\det (1 - t \Phi_F(X))^{2N} \det (1 - t \bar{\Phi}_F(X))^{2N}}. \]

Refine using \((S)U(2N)\) global fugacities. Example

\[ H_{SU(3)} = 1 + t^2 (\chi_{10001} + 1) + 2t^3 \chi_{00100} + t^4 (\chi_{01010} + \chi_{10001} + \chi_{20002} + 1) + \ldots \]
Now $\tilde{SU}(N)$

$$H_{\tilde{SU}(N)} = \int d\eta_{\tilde{SU}(N)}(X) \frac{\det (1 - t^2 \Phi_{\text{Adj}}(X))}{\det (1 - t \chi_{\text{Flav}}^{10\ldots0} \otimes \Phi_{\bar{F}F}(X))},$$

What is the flavor symmetry group?
Now $\tilde{SU}(N)$

$$H_{\tilde{SU}(N)} = \int d\eta_{\tilde{SU}(N)}(X) \frac{\det (1 - t^2 \Phi_{\text{Adj}}(X))}{\det (1 - t^{\chi_{10\ldots0}} \otimes \Phi_{F\bar{F}}(X))},$$

What is the flavor symmetry group?

Mesons satisfy symmetry / antisymmetry relations depending on the parity of $N$. 
The Higgs Branch of SQCD

\[
\begin{align*}
\text{Even } N & \quad \tilde{\text{SU}}(N) & \text{SO}(2N) \\
\text{Odd } N & \quad \tilde{\text{SU}}(N) & \text{USp}(2N)
\end{align*}
\]
Examples

\[ H_{SU(3)} = 1 + 36t^2 + 40t^3 + 630t^4 + 1120t^5 + 7525t^6 + ... \]

\[ H_{S\bar{U}(3)} = 1 + 21t^2 + 20t^3 + 336t^4 + 560t^5 + 3850t^6 + ... \]

The mesons are symmetric

\[ H_{S\bar{U}(3)} = 1 + [2, 0, 0]_{C_3} t^2 + \left( [0, 0, 1]_{C_3} + [1, 0, 0]_{C_3} \right) t^3 \]
\[ + \left( 2 [0, 1, 0]_{C_3} + 2 [0, 2, 0]_{C_3} + [4, 0, 0]_{C_3} + 2 \right) t^4 + ... \]
Examples

\[ \mathcal{H}_{SU(4)} = 1 + 64t^2 + 2156t^4 + 49035t^6 + \ldots \]
\[ \mathcal{H}_{\tilde{SU}(4)} = 1 + 28t^2 + 1106t^4 + 24381t^6 + \ldots \]

The mesons are anti-symmetric

\[ \mathcal{H}_{(4, 8)} = 1 + [0, 1, 0, 0]_{D_4} t^2 + \left( 2 [0, 0, 0, 2]_{D_4} + 2 [0, 0, 2, 0]_{D_4} + 2 [0, 2, 0, 0]_{D_4} + 2 [2, 0, 0, 0]_{D_4} + [4, 0, 0, 0]_{D_4} + 2 \right) t^4 + \ldots \]
On a space-time manifold with non-trivial cycles, one can define gauge bundles with disconnected groups.
String Theory Realization

On a space-time manifold with non-trivial cycles, one can define gauge bundles with disconnected groups. Put the class $S$ theory

\[ \text{on } C \times \mathbb{R}^{1,2} \times S^1 \] with:

- Twisted punctures
- Twist along $S^1$. 
The twisted sector (alone) has been studied

[Mekareeya, Song, Tachikawa, 1212.0545]

The index for a sphere with 3 punctures is

\[ \tilde{i} = \sum_{\lambda} \prod_{i=1,2,3} \frac{\tilde{K}_{\Lambda_i}(a_i)\tilde{P}_\lambda(a_it^{\Lambda_i})}{\tilde{K}_\rho\tilde{P}_\lambda(t^\rho)} \]

\[ \implies \text{"TQFT" structure of superconformal index.} \]
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\]

\[ \Rightarrow \text{\"TQFT\" structure of superconformal index.} \]

Work in progress: combined sectors.
Conclusion

- Representation theory of principal extension allows to construct interesting Lagrangian $\mathcal{N} = 2$ SCFTs
- They have non freely generated CBs in general
- They are type A theories with orthogonal / symplectic global symmetries
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Thank you for your attention!
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