Universidad de Oviedo

## The Importance of Being Disconnected: Principal Extension Gauge Theories

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Work with Alessandro Pini and Diego Rodriguez-Gomez (1804.01108)

## Introduction

Gauge theories initially formulated using simple Lie algebras. Possible extensions include:

- Products of many Lie algebras / groups

[Gaiotto, 0904.2715]


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- Products of many Lie algebras / groups

[Gaiotto, 0904.2715]
- Global structure ( $\pi_{1}$ ) of the group


Figure 1: The weights of line operators of gauge theories with $\mathbf{g}=s u(2)$.

> [Aharony, Seiberg, Tachikawa, 1305.0318]

- What about the global structure $\pi_{0}$ of the group?


## Introduction

Recent interest on discrete gauging (recall Hanany's talk). Gauging of a discrete symmetry allows for new Coulomb branch geometries

[Argyres, Martone, 1611.08602]

## Introduction

Alternative approach : start from a disconnected gauge group.

In this work

- We consider a special class of non-connected groups
- We focus on 4d $\mathcal{N}=2$ SCFTs
- We look at local physics and use algebraic counting tools.


## Outline

## Principal Extension Groups

The Coulomb branch Index

The Higgs Branch of SQCD and the Global Symmetry

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## Principal Extension Groups



## Principal Extension Groups

$$
A_{N-1}
$$

P-


Definition of principal extension

$$
\widetilde{G}=G_{\text {adj }} \rtimes_{\varphi} \Gamma_{\text {outer }}
$$

## Definition

## Examples:

$$
\widetilde{\mathrm{SU}}(N)=\mathrm{SU}(N) \rtimes_{\varphi}\{1, \mathcal{P}\}
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\begin{gathered}
\widetilde{\mathrm{SU}}(N)=\operatorname{SU}(N) \rtimes_{\varphi}\{1, \mathcal{P}\} \\
\widetilde{\mathrm{SO}}(2 N)=\mathrm{SO}(2 N) \rtimes_{\varphi}\{1, \mathcal{P}\}=\mathrm{O}(2 N)
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$(X, 1)$

$(X, \mathcal{P})$

In the case of $\operatorname{SU}(N)$, this is related to complex conjugation:

$$
\bar{X}=A^{-1} \mathcal{P}(X) A, \quad A^{T}=(-1)^{N-1} A \quad \text { and } \quad \operatorname{det} A=1 .
$$

## Representations ( $\widetilde{\mathrm{SU}}(3)$ EXAmple)



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## Coulomb branch Hilbert Series

Recall the superconformal index for a theory with gauge group $G$ and some fundamental matter multiplets:

$$
\mathcal{I}=\int \mathrm{d} \eta_{G}(X) \operatorname{PE}\left[\sum_{i \in \text { multiplets }} f^{\mathcal{R}_{i}} \chi_{\mathcal{R}_{i}}(X)\right] ;
$$

where

$$
\begin{gathered}
f^{V}=-\frac{\sigma \tau}{1-\sigma \tau}-\frac{\rho \tau}{1-\rho \tau}+\frac{\sigma \rho-\tau^{2}}{(1-\rho \tau)(1-\sigma \tau)} \\
f^{\frac{1}{2} H}=\frac{\tau(1-\rho \sigma)}{(1-\rho \tau)(1-\sigma \tau)}
\end{gathered}
$$

## Coulomb branch Hilbert Series

In the limit

$$
\tau \rightarrow 0, \quad \rho \sigma=: t
$$

we have

$$
f^{V}=t, \quad f^{\frac{1}{2} H}=0 .
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This gives the Coulomb branch index [Gadde, Rastelli, Razamat, 1110.3740].

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$$
\mathcal{I}_{G}^{\text {Coulomb }}(t)=\int_{G} \mathrm{~d} \eta_{G}(X) \frac{1}{\operatorname{det}\left(1-t \Phi_{\mathrm{Adj}}(X)\right)},
$$

This is Molien's formula for the Hilbert series of invariants of the adjoint representation.

## What is a Hilbert Series

Let $R$ be a Noetherian graded ring with $R_{0}=\mathbb{C}$,

$$
R=\bigoplus_{n \in \mathbb{N}} R_{n} .
$$

The Hilbert series of $R$ is

$$
\operatorname{HS}(R, t)=\sum_{n \in \mathbb{N}} t^{n} \operatorname{dim}_{\mathbb{C}} R_{n} .
$$

Example :

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\operatorname{HS}(\mathbb{C}[x], t)=\frac{1}{1-t}
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For polynomial rings, the Hilbert series is always a rational function [Hilbert, 1890.xxxx]

## What is a Hilbert Series

If the ring is a complete intersection,

$$
\operatorname{HS}(\mathbb{C}[\text { Gens }] /(\text { Rels }), t)=\frac{\prod_{\text {Rels }}\left(1-t^{\operatorname{deg}(\text { Rels })}\right)}{\prod_{\text {Gens }}\left(1-t^{\operatorname{deg}(\text { Gens })}\right)}
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$$

Example:

$$
\operatorname{HS}\left(\mathbb{C}\left[x_{1}, x_{2}, x_{3}\right] /\left(x_{1} x_{2}-x_{3}^{2}\right), t\right)=1+3 t+5 t^{2}+\ldots=\frac{1-t^{2}}{(1-t)^{3}}
$$

## Basic Invariant Theory

Particular case: freely-generated ring

$$
\mathbb{C}[\mathbf{x}]^{G} \cong \mathbb{C}\left[\iota_{1}, \ldots, I_{m}\right], \quad H S\left(\mathbb{C}[\mathbf{x}]^{G}, t\right)=\frac{1}{\prod_{i=1}^{m}\left(1-t^{\operatorname{deg} I_{i}}\right)}
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$$

In general, Hironaka decomposition for invariant rings:

$$
\begin{gathered}
\mathbb{C}[\mathbf{x}]^{G} \cong \bigoplus_{j=1}^{p} J_{j} \mathbb{C}\left[I_{1}, \ldots, I_{m}\right] \\
H S\left(\mathbb{C}[\mathbf{x}]^{G}, t\right)=\frac{\sum_{j=1}^{p} t^{\operatorname{deg} J_{j}}}{\prod_{i=1}^{m}\left(1-t^{\operatorname{deg} I_{i}}\right)} .
\end{gathered}
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## Basic Invariant Theory

Consider a finite group $G$, and a (finite-dimensional complex) representation $V$. Consider the ring of invariants

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$$

For a large class of groups ("reductive"),

$$
\frac{1}{|G|} \sum_{g \in G} \longrightarrow \int_{G} \mathrm{~d} \eta_{G}(X) .
$$

## Weyl Integration Formula

For a class function $f$,

$$
\int_{\widetilde{\mathrm{SU}}(N)} \mathrm{d} \eta_{\widetilde{\mathrm{SU}}(N)}(X) f(X)=\frac{1}{2}\left[\int \mathrm{~d} \mu_{N}^{+}(z) f(z)+\int \mathrm{d} \mu_{N}^{-}(z) f\left(z^{\mathcal{P}}\right)\right]
$$

[Wendt, 1999]
with

$$
\mathrm{d} \mu_{N}^{+}(z)=\prod_{j=1}^{N-1} \frac{\mathrm{~d} z_{j}}{2 \pi i z_{j}} \prod_{\alpha \in R^{+}\left(A_{N-1}\right)}(1-z(\alpha))
$$

and
$N$ even: $\quad \mathrm{d} \mu_{N}^{-}(z)=\prod_{j=1}^{N / 2} \frac{\mathrm{~d} z_{j}}{2 \pi i z_{j}} \prod_{\alpha \in R^{+}\left(B_{N / 2}\right)}(1-z(\alpha))$.
$N$ odd: $\quad \mathrm{d} \mu_{N}^{-}(z)=\prod_{j=1}^{(N-1) / 2} \frac{\mathrm{~d} z_{j}}{2 \pi i z_{j}} \prod_{\alpha \in R^{+}\left(C_{(N-1) / 2}\right)}(1-z(\alpha))$.

## The Coulomb Index

Computation for $\mathrm{SU}(N)$ :

$$
\mathcal{I}_{\mathrm{SU}(N)}^{\text {Counb }}(t)=\frac{1}{\prod_{i=2}^{N}\left(1-t^{i}\right)},
$$

corresponds to

$$
\mathbb{C}\left[\phi_{i j}\right]^{\mathrm{SU}(N)} \cong \mathbb{C}\left[\operatorname{Tr}\left(\phi^{k}\right)_{k=2, \ldots, N}\right],
$$

polynomial ring without any relation.

## The Non-Freely Generated Coulomb Branch

Computation for $\widetilde{\mathrm{SU}}(N)$ :

$$
\mathcal{I}_{\mathrm{SU}(N)}^{\mathrm{Coulomb}}(t)=\frac{\sum_{k_{1}<\cdots<k_{r} \text { odd }} t^{k_{1}+\cdots+k_{r}}}{\prod_{i \text { even }}\left(1-t^{i}\right) \prod_{i \text { odd }}\left(1-t^{2 i}\right)},
$$

Why?

$$
\operatorname{Tr}\left(\left(\phi^{\mathcal{P}}\right)^{k}\right)=(-1)^{k} \operatorname{Tr}\left(\phi^{k}\right) .
$$

There are "holes" in the structure of invariants.

## The Non-Freely Generated Coulomb Branch

Invariant theory interpretation:

1. The primary invariants $I_{k}$ for $2 \leq k \leq N$ defined by

$$
I_{k}= \begin{cases}\operatorname{Tr}\left(\phi^{k}\right) & \text { for } k \text { even } \\ \operatorname{Tr}\left(\phi^{k}\right)^{2} & \text { for } k \text { odd }\end{cases}
$$

2. The secondary invariants

$$
J_{k_{1}, \ldots, k_{r}}=\prod_{i=1}^{r} \operatorname{Tr}\left(\phi^{k_{i}}\right)
$$

for $k_{1}, \ldots, k_{r}$ odd and $3 \leq k_{1}<\cdots<k_{r} \leq N$, with $r$ even $(r=0$ corresponds to the trivial invariant 1).
Relations (among others):

$$
J_{k_{1}, \ldots, k_{r}}^{2}-I_{k_{1}} \ldots I_{k_{r}}=0
$$

## The Non-Freely Generated Coulomb Branch

For $N \geq 5$ the Coulomb branch of $\mathcal{N}=2 \widetilde{\mathrm{SU}}(N)$ gauge theories (of genus 0 ) is not freely generated.

See also [Argyres, Martone, 1804.03152]
[Bourton, Pini, Pomoni, 1804.05396]

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Example for $N=5$ : Coulomb branch parametrized by $I_{2}, I_{3}, I_{4}, I_{5}, J_{3,5}$ with relation $J_{3,5}^{2}=I_{3} / 5$.

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Example for $N=5$ : Coulomb branch parametrized by $I_{2}, I_{3}, I_{4}, I_{5}, J_{3,5}$ with relation $J_{3,5}^{2}=I_{3} I_{5}$.

Complex singularity of complex dimension 2 parametrized by $I_{2}$ and $I_{4}$.

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Consider the $\mathcal{N}=2 \operatorname{SU}(N)$ gauge theory with $2 N$ fundamental hypers.

$$
W \sim \operatorname{Tr} \tilde{Q} \phi Q \quad \Longrightarrow \quad Q \tilde{Q}-\frac{1}{N}(\operatorname{Tr} Q \tilde{Q}) \mathbf{1}_{N}=0
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$$

Higgs branch Hilbert series: [Hanany, Feng, He, Mekareeya, Benvenuti,...]

$$
H_{\mathrm{SU}(N)}=\int \mathrm{d} \eta_{\mathrm{SU}(N)}(X) \frac{\operatorname{det}\left(1-t^{2} \Phi_{\mathrm{Adj}}(X)\right)}{\operatorname{det}\left(1-t \Phi_{\mathrm{F}}(X)\right)^{2 N} \operatorname{det}\left(1-t \Phi_{\overline{\mathrm{F}}}(X)\right)^{2 N}} .
$$

Refine using $(S) U(2 N)$ global fugacities. Example

$$
H_{\mathrm{SU}(3)}=1+t^{2}\left(\chi_{10001}+1\right)+2 t^{3} \chi_{00100}+t^{4}\left(\chi_{01010}+\chi_{10001}+\chi_{20002}+1\right)+\ldots
$$

## Higgs branch Hilbert Series

Now $\widetilde{S U}(N)$

$$
H_{\widetilde{S U}(N)}=\int \mathrm{d} \eta_{\widetilde{\mathrm{SU}}(N)}(X) \frac{\operatorname{det}\left(1-t^{2} \Phi_{\mathrm{Adj}}(X)\right)}{\operatorname{det}\left(1-t \chi_{\left.\chi_{10, \ldots 0}^{\mathrm{Flav}} \otimes \Phi_{\mathrm{FF}}(X)\right)},\right.}
$$

What is the flavor symmetry group?

## Higgs branch Hilbert Series

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$$

What is the flavor symmetry group?

Mesons satisfy symmetry / antisymmetry relations depending on the parity of $N$.

## The Higgs Branch of SQCD



## Examples

$$
\begin{aligned}
& \mathcal{H}_{S U(3)}=1+36 t^{2}+40 t^{3}+630 t^{4}+1120 t^{5}+7525 t^{6}+\ldots \\
& \mathcal{H}_{\widetilde{S U}(3)}=1+21 t^{2}+20 t^{3}+336 t^{4}+560 t^{5}+3850 t^{6}+\ldots
\end{aligned}
$$

The mesons are symmetric

$$
\begin{aligned}
\mathcal{H}_{\widetilde{S U}(3)}= & 1+[2,0,0]_{c_{3}} t^{2}+\left([0,0,1]_{c_{3}}+[1,0,0]_{c_{3}}\right) t^{3} \\
& +\left(2[0,1,0]_{c_{3}}+2[0,2,0]_{c_{3}}+[4,0,0] c_{c_{3}}+2\right) t^{4}+\ldots
\end{aligned}
$$

## Examples

$$
\begin{aligned}
& \mathcal{H}_{S U(4)}=1+64 t^{2}+2156 t^{4}+49035 t^{6}+\ldots \\
& \mathcal{H}_{\widetilde{S U}(4)}=1+28 t^{2}+1106 t^{4}+24381 t^{6}+\ldots
\end{aligned}
$$

The mesons are anti-symmetric

$$
\begin{aligned}
\mathcal{H}_{(4,8)}= & 1+[0,1,0,0]_{D_{4}} t^{2}+\left(2[0,0,0,2]_{D_{4}}+2[0,0,2,0]_{D_{4}}\right. \\
& \left.+2[0,2,0,0]_{D_{4}}+2[2,0,0,0]_{D_{4}}+[4,0,0,0]_{D_{4}}+2\right) t^{4} \\
& +\ldots
\end{aligned}
$$

## String Theory Realization

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On a space-time manifold with non-trivial cycles, one can define gauge bundles with disconnected groups. Put the class $\mathcal{S}$ theory

on $\mathcal{C} \times \mathbb{R}^{1,2} \times S^{1}$ with:

- Twisted punctures
- Twist along $S^{1}$.


## String Theory Realization

The twisted sector (alone) has been studied [Mekareeya, Song, Tachikawa, 1212.0545]

The index for a sphere with 3 punctures is

$$
\tilde{I}=\sum_{\lambda} \prod_{i=1,2,3} \frac{\tilde{K}_{\Lambda_{i}}\left(\mathbf{a}_{i}\right) \tilde{P}_{\lambda}\left(\mathbf{a}_{i} i^{\Lambda_{i}}\right)}{\tilde{K}_{\rho} \tilde{P}_{\lambda}\left(t^{\rho}\right)}
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$\Longrightarrow$ "TQFT" structure of superconformal index.

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$$

$\Longrightarrow$ "TQFT" structure of superconformal index.

Work in progress : combined sectors.

## Conclusion

- Representation theory of principal extension allows to construct interesting Lagrangian $\mathcal{N}=2$ SCFTs
- They have non freely generated CBs in general
- They are type A theories with orthogonal / symplectic global symmetries


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Thank you for your attention!

