Localization of 4d $\mathcal{N} = 1$ Gauge Theories and Dualities

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Kimura, Nian, Zhao, work in progress

Motivation

- $\mathcal{N} = 1$ AGT for class $\mathcal{S}_k$ theories ('17 Mitev, Pomoni)
- Superconformal index of Argyres-Douglas theories ('16 Maruyoshi, Song)

$\Rightarrow \mathcal{N} = 1$ partition function should exist.

But localization doesn't work for 4d $\mathcal{N} = 1$ on S^4!

- Result from indirect approach ('17 Gorantis, Minahan, Naseer)

Outline of this Talk

- Current status of 4d $\mathcal{N} = 1$ localization
- Localize $\mathcal{N} = 1$ gauge theories on $S^2 \times \mathbb{R}^2_\epsilon$
  - Construct SUSY
  - BPS equations and vortex solutions
  - 1-loop determinant via index theorem
- Test Seiberg duality
- Summary and prospect

Status of 4d $\mathcal{N}=1$ Localization

- $\mathcal{N}=1$ and $\mathcal{N}=2$ superconformal index
- '13 Closset, Shamir:
  $\mathcal{N}=1$ localization on $S^2 \times T^2$ and $S^3 \times S^1$
- '14 Nishioka, Yaakov:
  $\mathcal{N}=1$ localization on $S^1 \times \mathcal{M}_3$
- '17 Gorantis, Minahan, Naseer:
  $\mathcal{N}=1$ on S^4 indirectly by analytic continuation of dimensions
- '15 Fujimori, Kimura, Nitta, Ohashi:
  $\mathcal{N}=1$ on $\mathbb{R}^2_\epsilon \times T^2$

N=1 localization on S4?

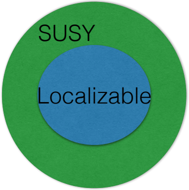
- '15 Terashima:
  $\nexists$ a Q-exact deformation, which is positive definite.
- '17 Lee:

$$\delta^2 = D_\theta + \frac{i}{\sin\theta} D_\varphi + \cdots$$

- '14 Knodel, Liu, Zayas:
  ('12 Dumitrescu, Festuccia, Seiberg)

$$\text{From old minimal model:} \quad D_\mu \epsilon =\sim \tilde{\epsilon}\,, \quad D_\mu \tilde{\epsilon} =\sim \epsilon$$

$$\text{From new minimal model:} \quad D_\mu \epsilon =\sim \epsilon\,, \quad D_\mu \tilde{\epsilon} =\sim \tilde{\epsilon}$$



Review $\mathcal{N}=1$ on $\mathbb{R}^2_\epsilon \times T^2$

('15 Fujimori, Kimura, Nitta, Ohashi)

- Metric:

$$ds^2 = |dz - iz(\epsilon dw + \bar{\epsilon} d\bar{w})|^2 + |dw|^2$$

- Vortices on $\mathbb{R}^2_\epsilon$
- Abelian partition function:

$$Z = \int_{C_a} \frac{d\sigma}{2\pi i \epsilon} \exp\left(-\frac{2\pi i \sigma \tau}{\epsilon}\right) \prod_{b=1}^{N} \Gamma\left(\frac{\sigma + m_b}{\epsilon}\right)$$

- Test 2d/4d correspondence ('11 Chen, Dorey, Hollowood, Lee)

Reminder of Localization Idean

S: $\mathcal{Q}$-invariant action with $\mathcal{Q}^2 = 0$; $h = \mathcal{Q}g$
t: dimensionless coupling constant

$$\frac{\partial}{\partial t}\int d\Phi e^{-S-th} = \int d\Phi(\mathcal{Q}g)e^{-S-t\mathcal{Q}g} = \int d\Phi\, \mathcal{Q}\left(ge^{-S-t\mathcal{Q}g}\right) = 0$$

- $t \to \infty$: the theory localizes to $h = 0$.
- The result of the classical (including instantons) and 1-loop contributions becomes exact:

$$Z = \int d\Phi\, e^{-S} = \int d(moduli)\, e^{-S(\Phi_{cl})} \frac{\det_{ferm}}{\det_{bos}}$$

- Localization Action: $h = \mathcal{Q}g$
  Localization Locus: Solutions to $h = 0$

Metric

$S^2 \times \mathbb{R}^2_\epsilon$:

$$ds^2 = \ell^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + |dw - iw\ell\varepsilon\, d\varphi|^2$$

Change of coordinates:

$$\tilde{\theta} = \theta\,, \quad \tilde{\varphi} = \varphi$$

$$z = w\, e^{-i\ell\varepsilon\varphi}\,, \quad \bar{z} = \bar{w}\, e^{i\ell\varepsilon\varphi}$$

$S^2 \times \mathbb{R}^2$:

$$ds^2 = \ell^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + |dz|^2$$

Strategy:

First construct SUSY on $S^2 \times \mathbb{R}^2$, then change coordinates to obtain SUSY on $S^2 \times \mathbb{R}^2_\epsilon$.

Killing Spinor

4d Killing spinor equation:

$$D_\mu \Upsilon = \frac{1}{2}\Gamma_\mu \Gamma_5 \Upsilon \,, \quad D_a \Upsilon = 0$$

with

$$\mu \in \{1, 2\} \,, \quad a \in \{3, 4\}$$

(In other dimensions: '12 & '13 Kawano, Matsumiya; '13 Lee, Yamazaki)

Decomposition:

$$\Upsilon = \epsilon \otimes \zeta_+ + \tilde{\epsilon} \otimes \zeta_-$$

where

$$\zeta_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,, \quad \zeta_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

ϵ and $\tilde{\epsilon}$ are 2d Killing spinors on S^2 satisfying

$$\nabla_\mu \epsilon = \frac{1}{2}\sigma_\mu \sigma_3 \epsilon \,, \quad \nabla_\mu \tilde{\epsilon} = -\frac{1}{2}\sigma_\mu \sigma_3 \tilde{\epsilon}$$

$\mathcal{N}=1$ SUSY on $\mathbb{R}^4$

(Notation from "Supergravity" by Freedman and Van Proeyen)

Vector multiplet:

$$\delta A_M = -\frac{1}{2}\bar{\Upsilon}\Gamma_M \Xi$$
$$\delta \Xi = \frac{1}{4}\Gamma^{MN} F_{MN}\Upsilon + \frac{i}{2}\Gamma_5 D\Upsilon$$
$$\delta D = \frac{i}{2}\bar{\Upsilon}\Gamma_5 \Gamma^M D_M \Xi$$

(Anti-)Chiral multiplet:

$$\delta\Phi = \frac{1}{\sqrt{2}}\bar{\Upsilon}P_L\Psi \qquad \delta\bar{\Phi} = \frac{1}{\sqrt{2}}\bar{\Upsilon}P_R\Psi$$
$$\delta P_L\Psi = \frac{1}{\sqrt{2}}P_L(\Gamma^M D_M\Phi + F)\Upsilon \qquad \delta P_R\Psi = \frac{1}{\sqrt{2}}P_R(\Gamma^M D_M\bar{\Phi} + \bar{F})\Upsilon$$
$$\delta F = \frac{1}{\sqrt{2}}\bar{\Upsilon}P_R\Gamma^M D_M\Psi - \bar{\Upsilon}P_R\Xi\Phi \qquad \delta\bar{F} = \frac{1}{\sqrt{2}}\bar{\Upsilon}P_L\Gamma^M D_M\Psi - \bar{\Upsilon}P_L\Xi\bar{\Phi}$$

$\mathcal{N}=1$ SUSY on $S^2 \times \mathbb{R}^2$

Vector multiplet:

$$\delta A_M = -\frac{1}{2}\bar{\Upsilon}\Gamma_M \Xi$$
$$\delta \Xi = \frac{1}{4}\Gamma^{MN} F_{MN}\Upsilon + \frac{i}{2}\Gamma_5 D \Upsilon$$
$$\delta D = \frac{i}{2}\bar{\Upsilon}\Gamma_5 \Gamma^M D_M \Xi$$

Chiral multiplet:

$$\delta \Phi = \frac{1}{\sqrt{2}}\bar{\Upsilon} P_L \Psi$$
$$\delta P_L \Psi = \frac{1}{\sqrt{2}} P_L (\Gamma^M D_M \Phi + F)\Upsilon + \frac{q}{\sqrt{2}}(P_L \Gamma^M D_M \Upsilon)\Phi$$
$$\delta F = \frac{1}{\sqrt{2}}\bar{\Upsilon} P_R \Gamma^M D_M \Psi - \bar{\Upsilon} P_R \Xi \Phi + \frac{q}{\sqrt{2}}(D_M \Upsilon)^T C_4 P_R \Gamma^M \Psi$$

$\mathcal{N}=1$ SUSY on $S^2 \times \mathbb{R}^2$

Anti-chiral multiplet:

$$\delta\bar{\Phi} = \frac{1}{\sqrt{2}} \bar{\Upsilon} P_R \Psi$$

$$\delta P_R \Psi = \frac{1}{\sqrt{2}} P_R (\Gamma^M D_M \bar{\Phi} + \bar{F}) \Upsilon + \frac{q}{\sqrt{2}} (P_R \Gamma^M D_M \Upsilon) \bar{\Phi}$$

$$\delta\bar{F} = \frac{1}{\sqrt{2}} \bar{\Upsilon} P_L \Gamma^M D_M \Psi - \bar{\Upsilon} P_L \Xi \bar{\Phi} + \frac{q}{\sqrt{2}} (D_M \Upsilon)^T C_4 P_L \Gamma^M \Psi$$

Assuming commuting Killing spinors:

$$\delta^2 = \mathcal{L}_\xi + \Lambda + \rho J_s^z$$

Back to $S^2 \times \mathbb{R}^2_\epsilon$ & turn on R- and flavor background gauge fields:

$$\delta^2 = -\frac{1}{\ell} \partial_\varphi - i\varepsilon (w \partial_w - \bar{w} \partial_{\bar{w}}) + (\sim \sin\theta\, \partial_z) - \frac{i}{2} (\epsilon - \frac{1}{\ell}) R + i\mathcal{F}$$

SUSY Actions

SUSY-exact Lagrangian:

$$\mathscr{L}_{\text{exact}} = \delta \mathcal{V}$$

$$\mathcal{V} = \mathcal{V}_{\text{gauge}} + \mathcal{V}_{\text{chiral}} + \mathcal{V}_{\text{H}}$$

$$\begin{aligned} \mathcal{V}_{\text{gauge}} &= \frac{1}{2g^2}(\delta\Xi)^\dagger \Xi \\ \mathcal{V}_{\text{chiral}} &= \frac{1}{2}\Big[(\delta P_L \Psi)^\dagger P_L \Psi + (\delta P_R \Psi)^\dagger P_R \Psi\Big] \\ \mathcal{V}_H &= \frac{i}{2}\Big[\Sigma^\dagger \Gamma_5 \Xi + \Xi^\dagger \Gamma_5 \widetilde{\Sigma}\Big] H(\Phi,\, \bar{\Phi}) \end{aligned}$$

SUSY-closed Lagrangian:

$$S_{\text{closed}} = 2 \int d^4x \, d\bar{\theta} \, d\theta \, i\tau_0 \Sigma$$

BPS Equations

$$
\begin{aligned}
\mathscr{L}^{\text{bosonic}}_{\text{exact}} = & \frac{1}{2g^2}\Big(F_{12} - \cos\theta\, F_{34} - g^2(|\Phi^I|^2 - \eta)\Big)^2 + \frac{\sin^2\theta}{2g^2}(F_{34})^2 \\
& + \frac{1}{2g^2}(F_{13})^2 + \frac{1}{2g^2}(F_{14})^2 + \frac{\cos^2\theta}{\ell^2}A_3^2 + \frac{\cos^2\theta}{\ell^2}A_4^2 \\
& + \frac{1}{2g^2}\left(F_{23} + \frac{1}{\ell}\sin\theta\, A_4\right)^2 + \frac{1}{2g^2}\left(F_{24} - \frac{1}{\ell}\sin\theta\, A_3\right)^2 \\
& + (D^{\bar{u}}\Phi^I)^\dagger(D_{\bar{u}}\Phi^I) + (D^{\bar{w}}\Phi^I)^\dagger(D_{\bar{w}}\Phi^I) + F^\dagger F + \frac{q^2}{\ell^2}(\Phi^I)^\dagger\Phi^I \\
& + (\text{total derivatives})
\end{aligned}
$$

BPS equations:

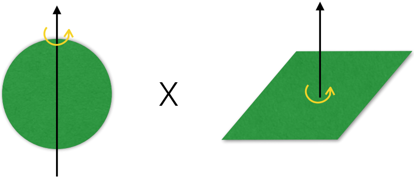
$$
F_{12} - \cos\theta\, F_{34} - g^2(|\Phi^I|^2 - \eta) = 0\,, \quad \sin\theta\, F_{34} = 0\,, \quad F_{13} = 0\,, \quad F_{14} = 0
$$

$$
F_{23} + \frac{1}{\ell}\sin\theta\, A_4 = 0\,, \quad F_{24} - \frac{1}{\ell}\sin\theta\, A_3 = 0\,, \quad \cos\theta\, A_3 = 0\,, \quad \cos\theta\, A_4 = 0
$$

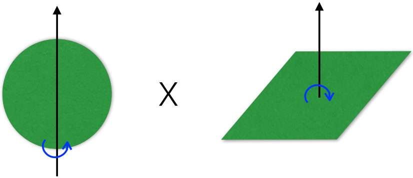
$$
D_{\bar{u}}\Phi^I = 0\,, \quad D_{\bar{w}}\Phi^I = 0\,, \quad F = 0
$$

Visualize Vortex Solutions

Vortex at the north pole and the origin:



Vortex at the south pole and the origin:



1-loop Det via Index Theorem

- Abelian chiral multiplet:

$$Z_{\text{chiral}}^{\text{1-loop}} = \prod_{p \geq 0} \prod_{k \geq 0} \frac{ip\epsilon + i(k+1)/\ell + im_J - im_I + in/\ell}{-ip\epsilon - ik/\ell + im_J - im_I + im/\ell}$$

$$= \frac{\Gamma_2(-m_J + m_I - m/\ell)}{\Gamma_2(1/\ell + m_J - m_I + n/\ell)}$$

m, n, k : vortex numbers at north pole, south pole and the origin

m_I, m_J : twisted masses

- Vector multiplet

$$Z_{\text{vec}}^{\text{1-loop}} = \prod_{p \geq 0} (p\epsilon + m_I - m/\ell)^{1/2} (p\epsilon + m_I - n/\ell)^{1/2}$$

$$= \frac{1}{\left(\Gamma\left(\frac{m_I - m/\ell}{\epsilon}\right)\Gamma\left(\frac{m_I - n/\ell}{\epsilon}\right)\right)^{1/2}}$$

Results

$$
\begin{aligned}
& Z_{\text{full}} \\
&= \sum_{m,\,n} Z^{\text{class}}_{\text{closed}}\, Z^{\text{1-loop}}_{\text{vec}}\, Z^{\text{1-loop}}_{\text{chiral}} \\
&= \sum_{m,\,n} \int_{C_I} d\sigma e^{-2\pi i \tau_0 (n-m)} \frac{1}{\left(\Gamma\left(\frac{\sigma - m/\ell}{\epsilon}\right)\Gamma\left(\frac{\sigma - n/\ell}{\epsilon}\right)\right)^{1/2}} \frac{\Gamma_2(-m_J + \sigma - m/\ell)}{\Gamma_2(1/\ell + m_J - \sigma + n/\ell)}
\end{aligned}
$$

where one picks up the pole at

$$\sigma = m_I\,.$$

Alternatively,

$$Z = Z^{\text{pert}} \cdot Z^{\text{non-pert}}$$

$$Z^{\text{pert}} = Z\big|_{m=n=0}$$

Compare with N=2 Nekrasov Partition Function

Perturbative part of $\mathcal{N} = 1$ adjoint chiral multiplet:

$$Z^{\text{pert}}_{\text{chiral, }\Omega\text{-bgd}} = \prod_{i<j} \frac{1}{\Gamma_2(\sigma_i - \sigma_j + \epsilon_1) \cdot \Gamma_2(\sigma_j - \sigma_i + \epsilon_1)}$$

Perturbative part of $\mathcal{N} = 1$ vector:

$$Z^{\text{pert}}_{\text{vec, }\Omega\text{-bgd}} = \prod_{i<j} \frac{1}{\Gamma\left(\frac{\sigma_i - \sigma_j}{\epsilon_2}\right) \cdot \Gamma\left(\frac{\sigma_j - \sigma_i}{\epsilon_2}\right)}$$

$$\Rightarrow Z^{\text{pert}}_{\text{chiral, }\Omega\text{-bgd}} \cdot Z^{\text{pert}}_{\text{vec, }\Omega\text{-bgd}} = \prod_{i<j} \left[\frac{1}{2\pi\epsilon_2} \cdot \frac{1}{\Gamma_2(\sigma_i - \sigma_j) \cdot \Gamma_2(\sigma_j - \sigma_i)} \right]$$

$$= Z^{\text{pert}}_{\text{N=2 vec, }\Omega\text{-bgd}}$$

Test Seiberg Duality

Similar to 2d Seiberg-like duality:

('12 Benini, Cremonesi; '12 Doroud, Gomis, Le Floch, Lee)

Test the original 4d Seiberg duality:

$$SU(N),\ N_F\ (> N) \Longleftrightarrow SU(N_F - N),\ N_F$$

By picking up different contours:

$$Z_{S^2 \times \mathbb{R}^2}(SU(N),\ N_F) = Z_{S^2 \times \mathbb{R}^2}(SU(N_F - N),\ N_F)$$

One can also test Kutasov-Schwimmer duality.

Summary and Prospect

Summary:

- SUSY on $S^2 \times \mathbb{R}^2$ and $S^2 \times \mathbb{R}^2_\epsilon$
- BPS equations and vortex solutions
- 1-loop determinant via index theorem
- Test Seiberg duality

Possible applications:

- Argyres-Douglas theories
- N=1 AGT
- Generalizations: loop operators, surface defects, quivers, $\cdots$

Thank you !