Topological recursion for Nekrasov partition functions through the AGT correspondence

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Overview

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   - $\mathcal{W}$-algebras and higher rank AGT
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   - $\mathcal{W}$-algebra from Miura transform of abelian currents
   - Generalized conformal Ward identities
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3. Computation of Nekrasov's instanton partition functions
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AGT correspondence

The AGT correspondence [Alday-Gaiotto-Tachikawa 2009] identifies 4d $\mathcal{N} = 2$ SYM with $SU(2)^{6G-3+M}$ in the Ω-background to 2d CFT with Virasoro symmetry over a Riemann surface $\Sigma_{G,M}$.

Gauge theory	**Liouville CFT** $c = 1 + 12Q^2$
mass parameters	external momenta
vevs of $SU(2)$ factors	internal momenta
bare coupling constants	cplx str moduli of $\Sigma_{G,M}$
Ω-deformation parameters $(\varepsilon_1, \varepsilon_2)$	Dispersion $\varepsilon = \sqrt{-\varepsilon_1 \varepsilon_2}$
	Background charge $Q = \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon}$
Instanton partition function	Conformal block

AGT dictionnary

$\mathcal{W}$-algebras and higher rank AGT

The AGT correspondence extends to higher rank [Wyllard 2009] identifying 4d $\mathcal{N}=2$ SYM with $SU(N)^{6G-3+M}$ in the Ω-background to 2d CFT with $\mathcal{W}$-symmetry ($\mathcal{W}_2 = Vir$) over a Riemann surface $\Sigma_{G,M}$.

Gauge theory	**Toda CFT** $c = N - 1 + 12Q^2$
mass parameters	external momenta
vevs of $SU(N)$ factors	internal momenta
bare coupling constants	cplx str moduli of $\Sigma_{G,M}$
Ω-deformation parameters $(\varepsilon_1, \varepsilon_2)$	Dispersion $\varepsilon = \sqrt{-\varepsilon_1\varepsilon_2}$ Background charge $Q = \frac{\varepsilon_1+\varepsilon_2}{\varepsilon}$
Instanton partition function	Conformal block

AGT dictionnary

Goal

Compute the instanton partition function corresponding to a conformal block of the form $\left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j) \right\rangle_{\Sigma_{G,M}}$

Method

Solve an infinite set of symmetry constraints called generalized conformal Ward identities (perturbatively in $\varepsilon \longrightarrow 0$ asymptotics).

This was already done for $N = 2$ [Chekhov-Eynard-Ribault 2013].

$\mathcal{W}$-algebra $\mathcal{W}_N$ from Miura transform of abelian currents

$$\begin{aligned} \mathbf{J} &= \partial\varphi & (1)\\ \mathbf{J}(\widetilde{x}\cdot E)\mathbf{J}(\widetilde{y}\cdot F) &\underset{x\sim y}{=} -\frac{(E,F)}{(x-y)^2} + \mathcal{O}(1) & (2)\\ \mathbf{J}(\widetilde{x}\cdot E)\, V_{\alpha_j}(z_j) &\underset{x\sim z_j}{=} -\frac{\alpha_j(E)}{x-z_j} V_{\alpha_j}(z_j) + \mathcal{O}(1) & (3)\\ \widehat{\mathcal{E}} = \sum_{k=0}^{N} (-1)^k \mathbf{W}^{(k)} \widehat{y}^{\,N-k} &= \,:(\widehat{y}-\mathbf{J}_1)\cdots(\widehat{y}-\mathbf{J}_N): & (4) \end{aligned}$$

φ is the Toda field, $\widehat{y} = Q\partial$, $\mathbf{J}_i = (h_i, \mathbf{J})$, h_i, $i = 1,..N$, are highest weights of the fundamental representation of $SU(N)$ and $\mathcal{W}_N$ is generated by the modes

$$\mathbf{W}^{(k)}(\widetilde{x}) = \sum_{n\in\mathbb{Z}} \frac{\mathbf{W}_n^{(k)}}{(x-x_0)^{n+k}} \qquad (5)$$

Quantum constraints

The form $Z = \int_{\mathfrak{X}} \mathcal{D}\varphi \exp\left(\frac{i}{\hbar} S[\varphi]\right)$ for the partition function of a quantum theory implies that for any set of observables $\{\mathcal{O}_I\}_I$ and any vector field $\frac{\partial}{\partial t} \in T\mathfrak{X}$,

$$\int_{\mathfrak{X}} \mathcal{D}\varphi \frac{\partial}{\partial t}\left(\exp\left(\frac{i}{\hbar}S[\varphi]\right)\prod_I \mathcal{O}_I(\varphi)\right) = 0 \tag{6}$$

$$\Leftrightarrow \quad \left\langle \frac{\partial S}{\partial t}\prod_I \mathcal{O}_I \right\rangle = i\hbar \sum_I \left\langle \dots \frac{\partial \mathcal{O}_I}{\partial t} \dots \right\rangle \tag{7}$$

$\hbar \longrightarrow 0$ yields classical equations of motion.

Generalized conformal Ward identities

$\mathcal{W}_N$ symmetry of Toda CFT then implies

GCWI

$\left\langle \mathbf{W}^{(k)}(\widetilde{x})\mathbf{J}_{a_1}(\widetilde{x}_1)\cdots\mathbf{J}_{a_n}(\widetilde{x}_n)\prod_{j=1}^{M} V_{\alpha_j}(z_j)\right\rangle_{\Sigma_{G,M}}$ is a holomorphic function of the variable $x \in \Sigma_{G,M} - \{z_1, \ldots, z_M, x_1, \ldots, x_n\}$ with singularities prescribed by OPE's.

- $n = 0$ case reduces to usual Ward identities.
- Miura transform yields recursive differential equations on multivalued

$$w_n\begin{pmatrix} a_1 & & a_n \\ x_1, & \ldots, & x_n \end{pmatrix} = \frac{\left\langle \mathbf{J}_{a_1}(\widetilde{x}_1)\cdots\mathbf{J}_{a_n}(\widetilde{x}_n)\prod_{j=1}^{M} V_{\alpha_j}(z_j)\right\rangle_{\Sigma_{G,M}}}{\left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j)\right\rangle_{\Sigma_{G,M}}} \tag{8}$$

Heavy charge regime

Rescaling $\alpha_j \longrightarrow \alpha_j/\varepsilon$ implies

$$\varepsilon \longrightarrow 0 \quad \Rightarrow \quad \alpha_j, Q \longrightarrow \infty \tag{9}$$

and we assume

Topological type asymptotics

$$w_n = \sum_{g=0}^{\infty} \varepsilon^{2g-2+n} \omega_{g,n} \tag{10}$$

GCWI decompose accordingly in powers of ε as recursive differential equations on w_n's.

$\omega_{g,n}$'s are computed by topological recursion

We have the topological recursion

$$\omega_{g,n+1}\left(\overset{i_0}{x_0}, X\right) = \sum_{\mu=1}^{N} \frac{1}{2\pi i} \oint_{x\in S_\mu} K_\mu(\overset{i_0}{x_0}, x) \Bigg(\omega_{g-1,n+2}\left(\overset{\mu+1}{x}, \overset{\mu}{x}, X\right) + \sum_{\substack{I\sqcup I'=X \\ h+h'=g}}^{\prime} \omega_{h,1+|I|}\left(\overset{\mu+1}{x}, I\right) \omega_{h',1+|I'|}\left(\overset{\mu}{x}, I'\right)\Bigg) \tag{11}$$

with integration contours S_μ's associated $\mu = 1, \ldots, N$ and K_μ satisfies

$$\left(Y_{\mu+1}(\widetilde{x}) - Y_\mu(\widetilde{x}) + Q\partial_x\right) K_\mu(\overset{i_0}{x_0}, \widetilde{x}) = \frac{1}{2}\int_{\overset{\mu}{x}}^{\overset{\mu+1}{x}} \omega_{0,2}(\overset{i_0}{x_0}, \cdot) \tag{12}$$

with $\left\langle \widehat{\mathcal{E}} \prod_{j=1}^{M} V_{\alpha_j}(z_j) \right\rangle \underset{\varepsilon\to 0}{\longrightarrow} (\widehat{y} - Y_1)\cdots(\widehat{y} - Y_N)$

Quantum spectral curve

$$\mathcal{E} = (\widehat{y} - Y_1) \cdots (\widehat{y} - Y_N) \tag{13}$$

is called the quantum spectral curve of the theory

$$E(x, y) = \underset{Q \to 0}{\text{Symb}}\ \mathcal{E}(x, \widehat{y}) \tag{14}$$

is the spectral curve of a classical integrable system.

Deformation of vertex operators

Vertex operators are related to the Toda field by

$$V_\alpha(z) =: \exp\left((\alpha, \varphi(z))\right) : \tag{15}$$

which implies

$$\frac{\partial}{\partial z_i} V_{\alpha_j}(z_j) = \delta_{i,j} : (\alpha_j, \mathbf{J}(z_j))\, V_{\alpha_j}(z_j) : \tag{16}$$

$$\frac{\partial}{\partial \alpha_i} V_{\alpha_j}(z_j) = \delta_{i,j} : \varphi(z_j) V_{\alpha_j}(z_j) : \tag{17}$$

Seiberg-Witten special geometry relations

$$\frac{\partial}{\partial z_i} \ln \left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j) \right\rangle_{\Sigma_{G,M}} = \frac{\left\langle : (\alpha_i, \mathbf{J}(z_i)) V_{\alpha_i}(z_i) : \prod_{\substack{j=1 \\ j\neq i}}^{M} V_{\alpha_j}(z_j) \right\rangle}{\left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j) \right\rangle} = \mathrm{ev}_{(z_i \cdot \alpha_i)} w_1 \quad (18)$$

$$\frac{\partial}{\partial \alpha_i} \ln \left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j) \right\rangle_{\Sigma_{G,M}} = \frac{\left\langle : \varphi(z_i) V_{\alpha_i}(z_i) : \prod_{\substack{j=1 \\ j\neq i}}^{M} V_{\alpha_j}(z_j) \right\rangle}{\left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j) \right\rangle} = \int_{reg}^{z_i} w_1(\overset{*}{x}) dx \quad (19)$$

Conclusions

Toplogical recursion solves generalized Ward identities of Toda CFT in the presence of topological heavy charge asymptotics.

$$w_n = \sum_{g=0}^{\infty} \varepsilon^{2g-2+n} \omega_{g,n} \tag{20}$$

Seiberg-Witten relations determine the corresponding conformal block.

$$\delta \ln \left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j) \right\rangle = \int_{\delta^*} w_1 \qquad \text{and} \qquad \delta w_n = \int_{\delta^*} w_{n+1} \tag{21}$$

The AGT correspondence then yields the corresponding Nekrasov instanton partition function.

Outlooks

- Apply the method to 4d $\mathcal{N} = 1$ AGT with Elli Pomoni (DESY)

- Apply the method to 5d $\mathcal{N} = 2$ AGT with Nezhla Aghaee (Uni Bern)

Thank you for your attention !