Supersymmetric Sigma Models: Dualities and Quantum Geometry

2nd international conference on:

Supersymmetric theories, dualites and deformations

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H. Jockers, U. Ninad, AG, Nuclear Physics B, arXiv: 1803.10253
H. Jockers, AG, Journal of Geometry and Physics, arXiv: 1505.00099

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Plan for Today

- Part I: Introduction
- Part II: Quantum cohomology from correlation functions
- Part III Duality and sphere partition function

Part I:

Introduction

Gauged linear sigma models (GLSMs)

- GLSMs are two-dimensional gauge theories with

Witten, 93

  1. $\mathcal{N} = (2,2)$ supersymmetry
  2. classical R-symmetry $U(1)_V \times U(1)_A$
  3. vector multiplets (gauge fields) and chiral multiplets (matter)

- Specific model determined by

  1. Gauge group $G = U(1)^\ell \times G_{\text{semi-simple}}$
  2. Matter spectrum

	G-rep.	R-charge	twisted mass
Φ_α	ρ_α	$\mathfrak{q}_\alpha$	$\mathfrak{m}_\alpha$

Moduli and Target Space Geometry

- GLSMs come as families of theories with moduli

1. Kähler m. $\tau = r - i\frac{\theta}{2\pi}$ (for each $U(1)$ factor in G)
2. Complex structure m. z (in superpotential)

Moduli and Target Space Geometry

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1. Kähler m. $\tau = r - i\frac{\theta}{2\pi}$ (for each $U(1)$ factor in G)
2. Complex structure m. z (in superpotential)

- At low energies fields take values in target space $\mathcal{X}_{\tau,z}$ (space of gauge inequivalent vacua)

$$U_{\tau,z}(\phi) = \underbrace{\left|\text{D-terms}(\tau)\right|^2}_{\text{from gauge group + spectrum}} + \underbrace{\left|\text{F-terms}(z)\right|^2}_{\text{from superpotential}} + \ldots$$

$$\mathcal{X}_{\tau,z}(\phi) = {U_{\tau,z}^{-1}(0)}\big/{G}$$

General philosophy

Study $\mathcal{X}_{\tau,z}$ with gauge theory, in particular its moduli dependence

Axial Anomaly and Conformal Models

Witten, 93; Morrison, Plesser, 94

- Axial R-symmetry $U(1)_A$ is anomalous in general.
- In conformal models anomaly cancels due to

$$\rho_\alpha : G \to SL(V) \qquad \left(\sum \text{charges} = 0 \right)$$

Consequence:

IR theory is family of non-trival $\mathcal{N} = (2,2)$ SCFT with central charge

$$c = -3 \dim \mathfrak{g} + 3 \sum_\alpha (1 - \mathfrak{q}_\alpha) \dim \rho_\alpha$$
$$= 3 \dim_{\mathbb{C}} \mathcal{X}$$

Example

- Consider GLSM with gauge group $G = U(1)$ and chiral matter spectrum

Chiral multiplet	$G = U(1)$ charge	$\mathfrak{q}_\alpha$	$\mathfrak{m}_\alpha$
P	-5	2	0
ϕ_a , $1 \leq a \leq 5$	$+1$	0	0

Witten, 93

- We find:

1. No anomaly: $\sum \text{charges} = 0$
2. Central charge: $c = -3 \dim \mathfrak{g} + 3 \sum_\alpha (1 - \mathfrak{q}_\alpha) \dim \rho_\alpha = 9$
3. Target space: $\mathcal{X} = \mathbb{CP}^4 \cap \{\text{homogeneous degree 5 poly.} = 0\}$
   $= \mathbb{CP}^4[5]$
   $=$ Quintic Calabi–Yau threefold

Part II:

Quantum cohomology from correlation functions

Correlation Functions of Adjoint Scalars

- Study correlation functions of scalars σ in vector multiplet V on A-twisted two-sphere with off shell supergravity background

Festuccia, Seiberg, 11; Closset et al., 15: Benini et al., 15

$$\text{BRST closed insertions:} \quad \sigma_N = \sigma(\text{North-pole})$$
$$\sigma_S = \sigma(\text{South-pole})$$

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- Quantum exact expression by localization:

Hori, Vafa, 00; Pestun, 07; Closset et al., 15: Benini et al., 15

$$\langle \sigma_N^n \sigma_S^m \rangle (Q) = \sum_{k \in \Lambda} Q^k \operatorname{Res}_\sigma \left[\left(\sigma - \epsilon \, \frac{k}{2} \right)^n \left(\sigma + \epsilon \frac{k}{2} \right)^m Z_{\text{1-loop}}(\sigma, k) \right]$$

with $\Lambda =$ magnetic charge lattice and $Q = \exp(-2\pi\tau)$

- Correlators $\langle \sigma_N^n \sigma_S^m \rangle = \langle \sigma_N^n \sigma_S^m \rangle (Q)$ are rational in Q Morrison, Plesser, 94

Correlator Relations

- **Aim:** Determine universal linear relations between correlators, such that with polynomials c_m and for all n

$$R_S = \sum_{m=0}^{M} c_m(Q) \langle \sigma_N^n \sigma_S^m \rangle(Q) = 0$$

- **Result:** Algorithm for their derivation

1. Gauge group G and matter spectrum immediately determine rational functions g_ℓ (for which we have a closed formula)
2. Find polynomials p_ℓ subject to constraint $\sum_{\ell=0}^{s} p_\ell(w) \cdot g_\ell(w) = 0$
3. $R_S = \sum_{\ell=0}^{s} Q^\ell \langle \sigma_N^n \, p_\ell(\sigma_S + \epsilon \, \ell) \rangle$ is universal correlator relation

Hilbert Space Interpretation

- Interpret correlators as matrix elements of operators $\boldsymbol{\sigma}_{\mathbf{N}}$ and $\boldsymbol{\sigma}_{\mathbf{S}}$ in gauge theory ground state $|\Omega\rangle_Q$

$$\langle\sigma_N^n\,\sigma_S^m\rangle(Q) = {}_Q\langle\Omega|\,\boldsymbol{\sigma}_{\mathbf{N}}^n\,\boldsymbol{\sigma}_{\mathbf{S}}^m\,|\Omega\rangle_Q$$

- **Consequences / results**:

1. Relations become operators $\boldsymbol{R}_{\mathbf{S}}$ that annihilate $|\Omega\rangle_Q$

$$R_S = {}_Q\langle\Omega|\,\boldsymbol{\sigma}_{\mathbf{N}}^n \cdot \boldsymbol{R}_{\mathbf{S}}\,|\Omega\rangle_Q = 0 \quad\Rightarrow\quad \boldsymbol{R}_{\mathbf{S}}\,|\Omega\rangle_Q = 0$$

2. Non-trival commutation relation between $\boldsymbol{Q}$ and $\boldsymbol{\sigma}_{\mathbf{S}}$:

$$\boldsymbol{\sigma}_{\mathbf{S}}^n \cdot \boldsymbol{R}_{\mathbf{S}}\,|\Omega\rangle_Q = 0 \quad\Rightarrow\quad [\boldsymbol{\sigma}_{\mathbf{S}}, \boldsymbol{Q}] = \epsilon\boldsymbol{Q}$$

$\Rightarrow$ Differential representation:

$$\boldsymbol{Q} = Q\,, \quad \boldsymbol{\sigma}_{\mathbf{S}} = \epsilon Q\partial_Q = \epsilon\Theta$$

Example: Part 1

- Recall: GLSM with quintic $\mathbb{CP}^4[5]$ target space

Chiral multiplet	$G = U(1)$ charge	$\mathfrak{q}_\alpha$	$\mathfrak{m}_\alpha$
P	-5	2	0
$\phi_a\ , 1 \leq a \leq 5$	1	0	0

- Three steps to determine universal correlator relation:

1. Spectrum gives: $g_0(w) = 1\ ,\quad g_1(w) = -\frac{w^4}{5} \cdot \prod_{s=1}^{4} \frac{1}{5w - \epsilon s}$
2. Solve constraint: $p_0(w) = w^4,\quad p_1(w) = +5 \prod_{s=1}^{4} (5w - \epsilon s)$
3. Write down relation:

$$R_S = \langle \sigma_N^n \sigma_S^4 \rangle + 5Q \langle \sigma_N^n \left(5\sigma_S + \epsilon\right) \cdots \left(5\sigma_S + 4\epsilon\right) \rangle = 0$$

Example: Part 2

- In differential representation R_S corresponds to operator $\boldsymbol{R}_{\mathbf{S}}$,

$$R_S = \langle \sigma_N^n \sigma_S^4 \rangle + 5Q \langle \sigma_N^n (5\sigma_S + \epsilon) \cdots (5\sigma_S + 4\epsilon) \rangle$$

$$\epsilon^{-4} \boldsymbol{R}_{\mathbf{S}} = \mathcal{L} = \Theta^4 + 5Q(5\Theta + 1)(5\Theta + 2)(5\Theta + 3)(5\Theta + 4)$$

This is the Picard–Fuchs operator of the quintic $\mathbb{CP}^4[5]$!

Intermediate summary of results — Jockers, Ninad, AG, 18

* Algorithm to determine correlator relations directly from defining gauge theory data
* Relations correspond to operators that describe the ground state's moduli dependence
* These are Picard–Fuchs (or GKZ) operators of the target space geometry and thus govern the target space quantum cohomology

Differential Operators in Terms of Correlators

- **Turn logic around:**
  Assume there exists an operator $\boldsymbol{R}_{\mathbf{S}}$ which annihaltes $|\Omega\rangle_Q$, then

$$\boldsymbol{R}_{\mathbf{S}} = \sum_{m=0}^{N} c_m(\boldsymbol{Q})\, \boldsymbol{\sigma}_{\mathbf{S}}^m \quad \Rightarrow \quad 0 = R_S = \sum_{m=0}^{N} c_m(Q)\, \langle \sigma_N^n \sigma_S^m \rangle \quad \forall n$$

- **Simple idea:**
  - * Fix order N and number of Qs (corresponding to $\dim_{\mathbb{C}} \mathcal{X}$ and $h^{1,1}$)
  - * List constraint for several n and solve for c_m in terms of $\langle \sigma_N^a \sigma_S^b \rangle$

Result: Jockers, Ninad, AG, 18

For several classes (fixed $\dim_{\mathbb{C}} \mathcal{X}$ and $h^{1,1}$) of Calabi–Yau manifolds:
Universal formulas for Picard–Fuchs operators in terms of gauge theory correlators

Example: One-Parameter Calabi–Yau Threefolds

For all 3-dim. Calabi–Yau manifolds with a single Kähler parameter:

$$\begin{aligned}\mathcal{L} = &+ \kappa_{0,3}^2(\epsilon\Theta)^4 - 2\kappa_{0,3}\,(\epsilon\Theta\,\kappa_{0,3})\,(\epsilon\Theta)^3 \\ &+ \Big[2\,(\epsilon\Theta\,\kappa_{0,3})^2 - \kappa_{0,3}\,\big(\epsilon^2\Theta^2\,\kappa_{0,3} + \kappa_{2,3}\big)\Big](\epsilon\Theta)^2 \\ &+ \Big[2\kappa_{2,3}\,(\epsilon\Theta\,\kappa_{0,3}) - \kappa_{0,3}\,(\epsilon\Theta\,\kappa_{2,3})\Big](\epsilon\Theta) \\ &+ \Big[\kappa_{2,3}^2 - \kappa_{0,3}\kappa_{3,4} - (\epsilon\Theta\,\kappa_{0,3})\,(\epsilon\Theta\,\kappa_{2,3}) + \kappa_{2,3}\,\big(\epsilon^2\Theta^2\,\kappa_{0,3}\big)\Big]\end{aligned}$$

where $\kappa_{n,m} = \langle \sigma_N^n \sigma_S^m \rangle$ are correlators of the associated GLSM

Jockers, Ninad, AG, 18

- Automatically fulfills constraint imposed by special geometry

$$c_1 = \frac{1}{2}c_2c_3 - \frac{1}{8}c_3^3 + \epsilon\Theta c_2 - \frac{3}{4}c_3\,(\epsilon\Theta c_3) - \frac{1}{2}\epsilon^2\Theta^2 c_3 \qquad \text{for } c_4 = 1$$

Strominger, 90
Almkvist, Zudilin, 04; van Enckevort, van Straten, 04

Part III:

Duality and sphere partition

Taking One Step Back

- **Approach so far:**
  1. GLSM is given
  2. Calculate RG-invariant observable to analyse low energy physics

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Taking One Step Back

- **Approach so far:**
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- **Ambitious question:** Can we do the reverse?
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- **More modest question:** Can we test Seiberg-like dualities of 2D gauge theories?

  Seiberg, 94: Hori, 11
  1. Take two conjecturally dual theories
  2. Compare IR observable, this talk: Two sphere partition function

e.g. Jockers et al., 12; Jockers, AG, 15; Closset et al., 17
Römelsberger, 06; Dolan, Osborne, 09

Skew Symplectic Sigma Models and Duality Proposal

- Skew symplectic sigma models $SSSM_{k,m,n}$ are class of GLSM with
  1. Gauge group: $G = U(1) \times USp(2k)$ $\Rightarrow$ 1 Kähler modulus τ
  2. Matter: Singlets + fundamentals, determined by m and n
- Conformal models with smooth CY_3 target space:

$$\begin{aligned} &SSSM_{1,12,6} && \text{with} \quad G = U(1) \times SU(2) \\ &SSSM_{2,9,6} && \text{with} \quad G = U(1) \times USp(4) \end{aligned}$$

Duality Proposal: Jockers, AG, 15

$SSSM_{1,12,6}$ and $SSSM_{2,9,6}$ are (strong-weak) dual to each other

Duality from Sphere Partition Function

- Both models have a strong and a weak coupling phase

$$\begin{aligned} \mathrm{Re}\,\tau \to +\infty &: \quad \text{weak coupling} \\ \mathrm{Re}\,\tau \to -\infty &: \quad \text{strong coupling} \end{aligned}$$

- Quantum exact formula for sphere partition function as sum of finite-dimensional Mellin–Barnes integral

Benini, Cremonesi, 12; Doroud et al., 12

$$Z_{S^2}(\tau) \sim \sum_{k \in \substack{\text{magnetic} \\ \text{charges}}} \int_{\mathfrak{h}} d^{\dim \mathfrak{h}}\sigma \; Z_{\text{class}}(\tau, \sigma, k) \; Z_{\text{1-loop}}(\sigma, k)$$

Strong evidence of (strong-weak) duality:

Jockers, AG, 15

$$Z_{S^2}(SSSM_{1,12,6}, \tau) = Z_{S^2}(SSSM_{2,9,6}, -\tau)$$

Summary and Outlook

- **Key results:**
  1. Detailed and explicit connection of GLSM correlators to target space quantum cohomology
  2. Universal formulas for Picard–Fuchs operators in terms of correlators
  3. Strong-weak coupling duality between pair of non-Abelian GLSMs

- **Room for future work:**
  1. More systematic understanding of dualities, especially for non-Abelian models
  2. Going backwards: Reconstruct GLSM from given a Picard–Fuchs operator

Thank you for your attention!

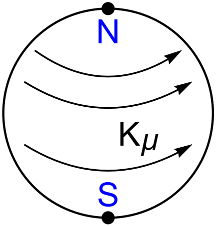
Questions?

Backup

A-twist and Q-dependence of Correlators

- Study correlation functions of scalars σ in vector multiplet V on A-twisted two-sphere with off shell supergravity background

Festuccia, Seiberg, 11; Closset et al., 15: Benini et al., 15



Killing vector field K_μ

Graviphoton background $C_\mu = \frac{\epsilon}{2} K_\mu$

- Dependence of correlators $\langle \sigma_N^n \sigma_S^m \rangle(Q)$ on Q determined by sum of matter charges $S_Q = \sum$ charges

Witten, 93; Morrison, Plesser, 94

$S_Q > 0$: polynomial in Q

$S_Q = 0$: rational in Q

$S_Q < 0$: polynomial in Q^{-1}

Rational functions g_ℓ from GLSM spectrum

- Starting point in derivation of correlator relations are rational functions g_ℓ
- They follow immediately from GLSM matter spectrum

❶

	G-rep.	R-charge	twisted mass
Φ_α	ρ_α	$\mathfrak{q}_\alpha$	$\mathfrak{m}_\alpha$

❷

$$g_\ell(w) = \prod_\alpha \frac{\prod_{s=1}^{\infty} \quad w \cdot \rho_\alpha + \mathfrak{m}_\alpha + \epsilon \left(1 - \frac{\mathfrak{q}_\alpha}{2} - s\right)}{\prod_{s=1+\rho_\alpha\cdot\ell}^{\infty} \quad w \cdot \rho_\alpha + \mathfrak{m}_\alpha + \epsilon \left(1 - \frac{\mathfrak{q}_\alpha}{2} - s\right)}$$

Geometric Interpretation of Correlators

- To geometric target space $\mathcal{X}$ associate Givental I-function $I_{\mathcal{X}}$
  1. Valued in vertical cohomology, $I_{\mathcal{X}}(Q,\epsilon) \in H^{\text{even}}(\mathcal{X})$
  2. Governs quantum cohomology of $\mathcal{X}$ / OPE of σ fields **Givental, 96**
  3. Generates correlators: $\langle \sigma_N^n \, \sigma_S^m \rangle = \int_{\mathcal{X}} (-\epsilon\Theta)^n I_{\mathcal{X}}(Q,-\epsilon) \cup (\epsilon\Theta)^m I_{\mathcal{X}}(Q,\epsilon)$

Ueda, Yoshida, 16; Kim et al., 16

- Differential operators $\boldsymbol{R}_{\mathbf{S}}$ annihilate Givental I-function

$$0 = R_S = \int_{\mathcal{X}} (-\epsilon\Theta)^n I_{\mathcal{X}}(Q,-\epsilon) \cup \left[\underbrace{\sum_m c_m \, (\epsilon\Theta)^m}_{\boldsymbol{R}_{\mathbf{S}}} \, I_{\mathcal{X}}(Q,\epsilon) \right]$$

Result

Differential operators $\boldsymbol{R}_{\mathbf{S}}$ annihilate Givental I-function, $\boldsymbol{R}_{\mathbf{S}} I_{\mathcal{X}} = 0$, and thus govern target space quantum cohomology

Skew Symplectic Sigma Models $SSSM_{k,m,n}$

- Class of skew symplectic sigma models $SSSM_{k,m,n}$
  1. Gauge group $G = U(1) \times USp(2k)$ $\Rightarrow$ 1 Kähler modulus τ
  2. Matter spectrum

Chiral multiplet	ρ_α	$\mathfrak{q}_\alpha$	$\mathfrak{m}_\alpha$
$P_{[ij]}$, $1 \leq i < j \leq n$	$\mathbf{1}_{-2}$	2	0
Q	$\mathbf{2k}_{-3}$	2	0
ϕ_a, $a = 1, \ldots, m$	$\mathbf{1}_2$	0	0
X_i, $i = 1, \ldots, n$	$\mathbf{2k}_1$	0	0

- Integers (k, m, n) determine axial anomaly and central charge

Conformal models with smooth CY_3 target space ($c = 9$)

$SSSM_{1,12,6}$ and $SSSM_{2,9,6}$

Superpotential and Target Space

- Superpotential: Most general holomorphic, gauge invariant function of R-charge 2, here

$$W_z = \mathrm{tr}\left[P\,A(\phi) + P\,X^T \epsilon\, X\right] + B(\phi) Q^T \epsilon\, X$$

with $A(\phi)_{[ij]} = A^a_{[ij]}\phi_a, \quad B(\phi)_i = B^a_i \phi_a, \quad \epsilon = \begin{pmatrix} 0 & \mathbf{1}_{k\times k} \\ -\mathbf{1}_{k\times k} & 0 \end{pmatrix}$

- Complex structure moduli: $A^a_{[ij]}$ and B^a_i
- Determine semi-classical target space $\mathcal{X}_{\boldsymbol{\tau},z}$ in three steps:
  1. Write down D-terms and F-terms
  2. Solve $D(\tau) = F = 0$ for $\mathrm{Re}\,\tau \gg 0$ or $\mathrm{Re}\,\tau \ll 0$
  3. For consistency: No unbroken non-Abelian subgroup $G' \subset G$

$SSSM_{k,m,n}$ Phase $\operatorname{Re}\tau \gg 0$: Weak Coupling

- Solving $D(\tau) = F = 0$ for $\operatorname{Re}\tau \gg 0$ gives

Semi-classical target space variety $\operatorname{Re}\tau \gg 0$

$$\mathcal{X}_{k,m,n} = \left\{ \phi \in \mathbb{CP}^{m-1} \,\middle|\, \operatorname{rk} A(\phi) \leq 2k \text{ and } A(\phi) \cdot B(\phi) = 0 \right\}$$

- Heuristic (incomplete) derivation:

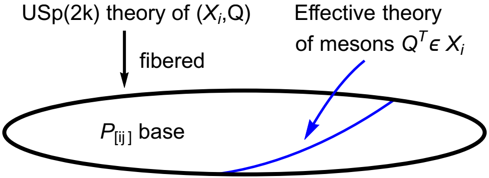
1. Abelian D-term: Not all $\phi_a = 0 \Rightarrow \phi \in \mathbb{CP}^{m-1}$
2. F-term for $P_{[ij]}$: $A(\phi) = -X^T \epsilon X$ with $X = (X_1, \ldots, X_n)$
   $\Rightarrow \operatorname{rk} A(\phi) \leq \operatorname{rk} X \leq 2k$
3. F-term for Q: $F_Q = \epsilon X \cdot B(\phi) = 0$
   $\Rightarrow -X^T F_Q = A(\phi) \cdot B(\phi) = 0$

$SSSM_{k,m,n}$ Phase $\mathrm{Re}\,\tau \ll 0$: Strong Coupling

- Gauge group broken to non-Abelian subgroup $USp(2k)$
  1. Strong coupling dynamics in infrared
  2. Dual description: Weakly coupled mesons

c.f. Hori, 11

- For $SSSM_{k,m,n}$:

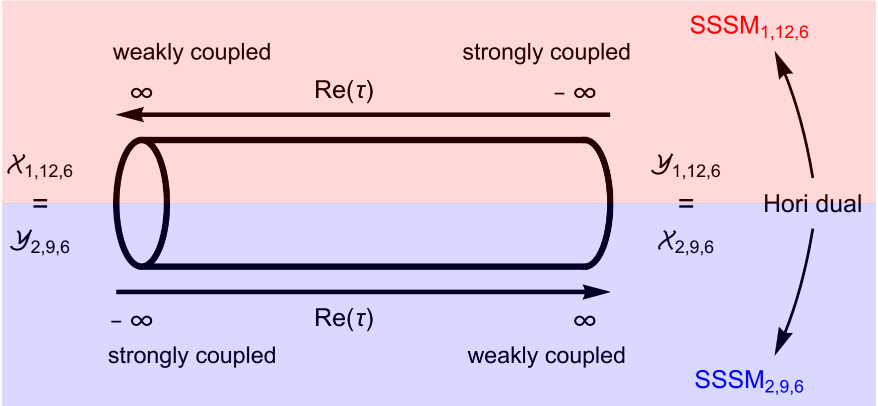


Semi-classical target space variety $\mathrm{Re}\,\tau \ll 0$

$$\mathcal{Y}_{k,m,n} \simeq \mathcal{X}_{\tilde{k},\tilde{m},n} \quad \text{with} \quad \tilde{k} = \frac{n}{2} - k, \quad \tilde{m} = \frac{n(n+1)}{2} - m, \quad n \text{ even}$$

In particular: 1) $\mathcal{Y}_{1,12,6} \simeq \mathcal{X}_{2,9,6}$ and 2) $\mathcal{Y}_{2,9,6} \simeq \mathcal{X}_{1,12,6}$

Phase Structure and Conjecture of Duality



Sphere Partition Function Supports Duality Proposal

Strong evidence of duality:

$$Z_{S^2}(SSSM_{1,12,6},\tau) = Z_{S^2}(SSSM_{2,9,6},-\tau)$$

- In addition we obtained:
  1. Geometric invariants of Calabi–Yau target spaces (from Kähler potential)

	$\mathcal{X}_{1,12,6} = \mathcal{Y}_{2,9,6}$	$\mathcal{Y}_{1,12,6} = \mathcal{X}_{2,9,6}$
Degree	33	21
Euler characterstic	-102	-102
Genus 0 GW invariants	252, 1854 ...	387, 4671 ...

  2. Combinatorial algorithm to evaluate Mellin–Barnes integrals of any dimension