

Surface operators and duality relations in $N=2$ gauge theories

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BERN

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This talk is mainly based on:

- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, V. Gupta, R.R. John and A. L., *"Surface operators, dual quivers and residues"*, [arXiv:1807.xxxxx](#)
- S.K. Ashok, S. Ballav, M. Billò, E. Dell'Aquila, M. Frau, R.R. John and A. L., *"Surface operators in conformal SQCD"*, [arXiv:180y.yyyyy](#)
- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, V. Gupta, R.R. John and A. L., *"Surface operators in 5d gauge theories and duality relations"*, **JHEP 1805 (2018) 046**, [arXiv:1712.06946](#)
- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, V. Gupta, R.R. John and A. L., *"Surface operators, chiral rings and localization in $N=2$ theories"*, **JHEP 1711 (2017) 137**, [arXiv:1707.08922](#)
- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, R.R. John and A. L., *"Modular and duality properties of surface operators in $N=2^*$ gauge theories"*, **JHEP 1707 (2017) 068**, [arXiv:1702.02833](#)

but it builds on **a very vast literature ...**

Plan of the talk

1. Introduction
2. Surface operators in $N=2$ gauge theories
3. Duality relations
4. Conclusions

Defects

- The study of how a QFT responds to the presence of **defects** has recently received a lot of attention.
- **Defects** are disturbances supported on sub-manifolds of the space-time that can be used to probe the theory in the bulk:
 1. Local operators ($d=0$), *e.g.* chiral correlators

$$\langle \text{Tr} \phi^{n_1}(x_1) \cdots \text{Tr} \phi^{n_k}(x_k) \rangle$$

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4. Domain walls ($d=3$)

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3. **Surface operators ($d=2$)**



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Defects

- There are different ways of describing defects in a $4d$ QFT:
 1. Modifying the QFT path-integral by imposing a **singular behavior** of the fields on the defect.
 2. Coupling the QFT to **additional degrees** of freedom localized on the defect.
 3. Using a String Theory realization of the QFT and introducing **extra D-branes or M-branes** to represent the defect.
 4. ...

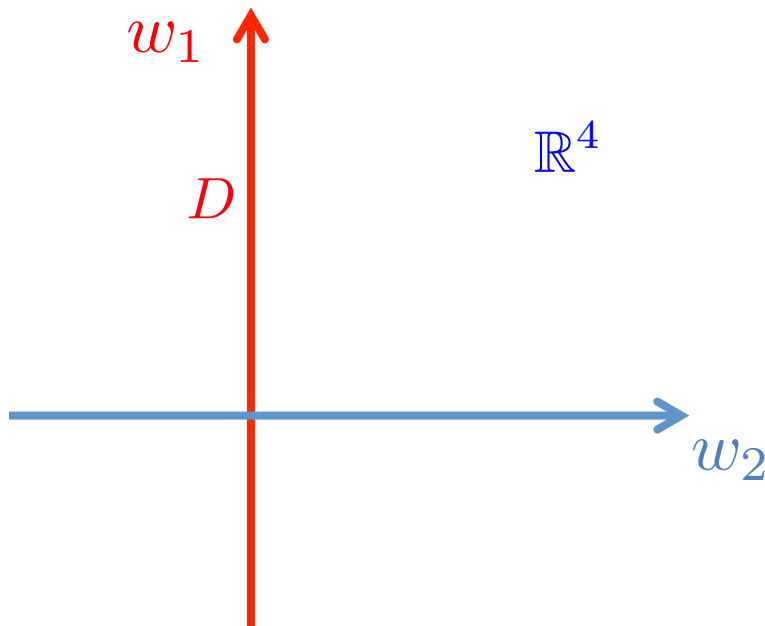
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Surface operators

Surface operators

- **Surface operators** in a $4d$ theory are operators supported on a $2d$ submanifold: $D \subset \mathbb{R}^4$
 - They are particular examples of non-local operators that play the role of “thermometers” for the QFT: when introduced in the path-integral, they provide us with valuable information (phases, non-perturbative features, ...) on the QFT. (Gukov+Kapustin '13)

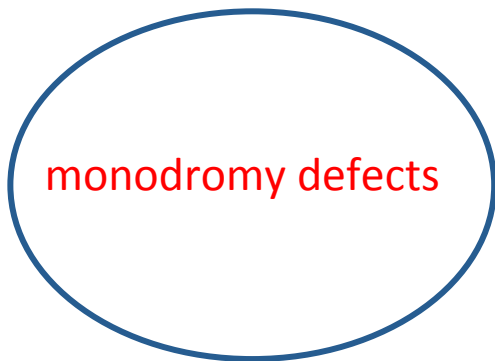


$$\mathbb{R}^4 \simeq \mathbb{C}^2 : (w_1, w_2)$$

$$D \simeq \mathbb{C} : (w_1, 0)$$

Surface operators

- Surface operators can be described:
 - as singularities (or monodromy defects) for the gauge field A_μ along the surface D in the $4d$ space-time



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 - as singularities (or monodromy defects) for the gauge field A_μ along the surface D in the $4d$ space-time
 - as coupled $2d$ - $4d$ systems, namely as a $2d$ theory supported on D with a flavor symmetry group G that is gauged in $4d$.



monodromy defects

coupled $2d$ - $4d$ systems

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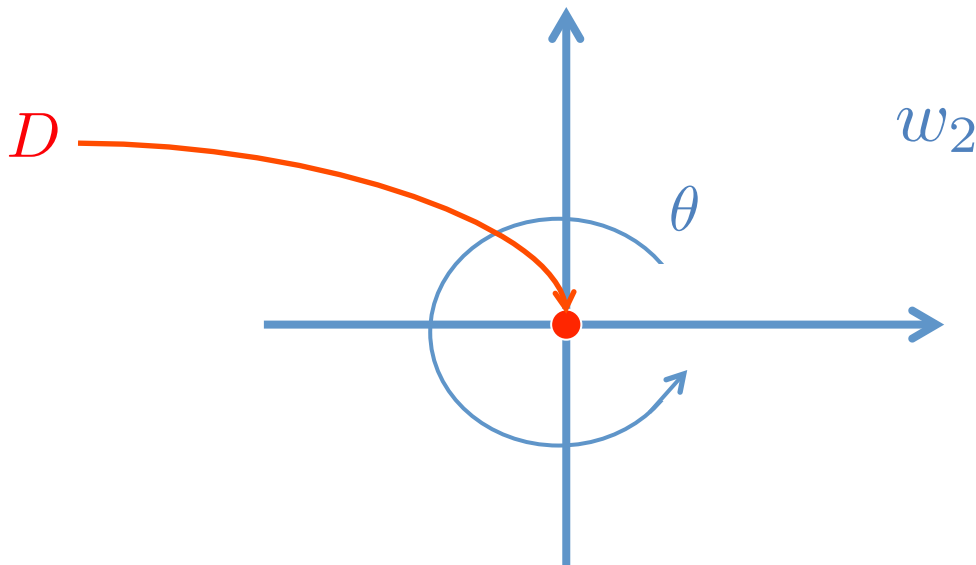
Surface operators as monodromy defects

- The presence of the defect can be encoded in a **singular behavior** of the $SU(N)$ gauge field on a loop around the defect:

$$A = -\text{diag} \left(\underbrace{\gamma_1, \dots, \gamma_1}_{n_1}, \underbrace{\gamma_2, \dots, \gamma_2}_{n_2}, \dots, \underbrace{\gamma_M, \dots, \gamma_M}_{n_M} \right) d\theta$$

where

$$\sum_i n_i = N \quad , \quad w_2 = \rho e^{i\theta} \quad \text{(Gukov+Witten '06, '08)}$$



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- This monodromy, *i.e.* the vector $\vec{n} = \{n_1, n_2, \dots, n_M\}$, breaks the gauge symmetry $SU(N)$ to a (Levi) subgroup \mathbb{L} , which characterizes the defect

$$SU(N) \rightarrow S[U(n_1) \times U(n_2) \times \dots \times U(n_M)]$$

Surface operators as monodromy defects

- In presence of the defect, quantized magnetic fluxes are allowed for each group $U(n_i)$ factor:

$$\frac{1}{2\pi} \int_D \text{Tr } F_{U(n_i)} = m_i \in \mathbb{Z} \quad , \quad \sum_i m_i = 0$$

- So besides the usual instanton factor

$$\exp \left[2\pi i \tau \left(\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{Tr } F \wedge F \right) \right]$$

complexified gauge coupling



in the path-integral we can insert also a new type of factor

$$\exp \left[2\pi i \sum_i \eta_i \left(\frac{1}{2\pi} \int_D \text{Tr } F_{U(n_i)} \right) \right]$$



constant “electric” parameters

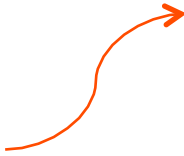
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- It can be shown that this implies that

$$\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{Tr } F \wedge F = k + \sum_i \gamma_i m_i$$

instanton number 

 new contribution due to the defect

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- Thus

$$-S_{\text{inst}} = 2\pi i \tau \left(\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{Tr } F \wedge F \right) + 2\pi i \sum_i \eta_i \left(\frac{1}{2\pi} \int_D \text{Tr } F_{U(n_i)} \right)$$

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- Thus

$$\begin{aligned} -S_{\text{inst}} &= 2\pi i \tau \left(k + \sum_i \gamma_i m_i \right) + 2\pi i \sum_i \eta_i m_i \\ &= 2\pi i \tau k + 2\pi i \sum_i (\eta_i + \tau \gamma_i) m_i \end{aligned}$$

Surface operators as monodromy defects

- The non-perturbative weight in the path-integral is then:

$$e^{2\pi i \tau k} \prod_i e^{2\pi i t_i m_i} \quad \text{with} \quad t_i = \eta_i + \tau \gamma_i$$

- So that the instanton partition function becomes

$$Z_{\text{inst}} = \sum_{\{k, m_i\}} \left(e^{2\pi i \tau k} \prod_i e^{2\pi i t_i m_i} \right) Z_{\{k, m_i\}}$$

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which, through a change of variables, takes the form

$$Z_{\text{inst}} = \sum_{\{d_i\}} q_i^{d_i} Z_{\{d_i\}}$$

where $d_i \in \mathbb{N}$ and

$$\left. \begin{aligned} q_1 &\sim e^{2\pi i t_1} , \\ q_i &\sim e^{2\pi i (t_i - t_{i-1})} , \quad i = 2, \dots, M-1 , \\ q_M &\sim e^{2\pi i \tau} e^{-2\pi i t_{M-1}} . \end{aligned} \right\} \Rightarrow q \sim e^{2\pi i \tau} = q_1 \cdots q_M$$

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← **ramified** instanton partition function

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Ramified instanton partition function

- Like the usual instanton partition function, also the ramified instanton partition function can be computed **using localization** (Kanno, Tachikawa '11)
- The idea is to consider the orbifold

$$\begin{array}{c} \mathbb{R}^4 \\ D \end{array} \begin{array}{l} \longrightarrow \\ \longrightarrow \end{array} \mathbb{C}_{\epsilon_1} \times \frac{\mathbb{C}_{\epsilon_2} \times \mathbb{C}}{Z_M} \times \mathbb{C} \times \mathbb{C}$$

in the presence of an Omega-background with parameters ϵ_1 and ϵ_2 to regulate the divergences and localize (Nekrasov '02,...)

- Introducing v.e.v's a for the adjoint scalars and going to a generic point in the Coulomb branch, we have


$$Z_{\text{inst}} = \sum_{\{d_i\}} q_i^{d_i} Z_{\{d_i\}}(a, \epsilon_1, \epsilon_2)$$

← explicitly calculable

Ramified instanton partition function

- From Z_{inst} we obtain

$$\log(1 + Z_{\text{inst}}) = -\frac{F_{\text{inst}}}{\epsilon_1 \epsilon_2} + \frac{W_{\text{inst}}}{\epsilon_1} + \dots$$




prepotential (twisted) superpotential

- The prepotential F describes the effective $4d$ dynamics
- The (twisted) superpotential W accounts for the effective dynamics on the $2d$ defect

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- Indeed
$$\frac{1}{\epsilon_1 \epsilon_2} \sim \text{Svol}(\mathbb{C}_{\epsilon_1 \epsilon_2}^2) \sim \int d^4x \, d^2\theta_1 \, d^2\theta_2$$

$$\frac{1}{\epsilon_1} \sim \text{Svol}(\mathbb{C}_{\epsilon_1}) \sim \int d^2x \, d\theta_1 \, d\bar{\theta}_1$$

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prepotential
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- For example, in the case $SU(2) \rightarrow S[U(1) \times U(1)]$, we have


$$Z_{\text{inst}} = \frac{q_1}{\epsilon_1(2a + \epsilon_1 + \epsilon_2)} + \frac{q_2}{\epsilon_1(-2a + \epsilon_1 + \epsilon_2)} + \dots$$

$$W_{\text{inst}} = \frac{q_1}{2a} - \frac{q_2}{2a} + \dots \quad \langle \phi \rangle = \text{diag}(a, -a)$$

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Can we calculate W in a different way
and obtain some physical intuition ?

Surface defect as a coupled $2d/4d$ system

- One considers a $4d$ gauge theory with group $G=SU(N)$
- One then couples it with a $(2,2)$ $2d$ GLSM with G as a global symmetry
(Gukov+Witten '06, ...
Gadde+Gukov '13,
Gaiotto+Gukov+Seiberg '13, ...)
- Simplest example:

CP^1 $2d$ σ – model with $SU(2)$ flavour symmetry

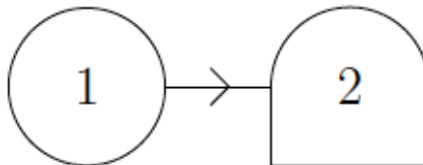
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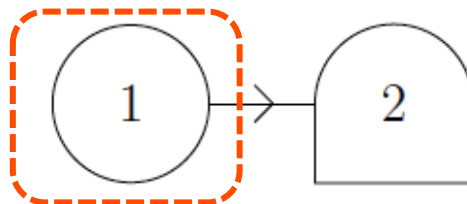
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$2d$ $U(1)$ gauge theory



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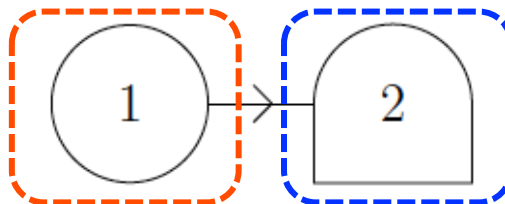
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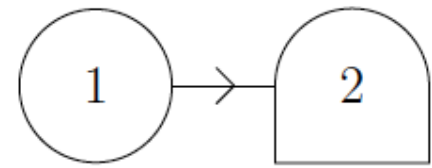
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$2d$ $U(1)$ gauge theory



$4d$ $SU(2)$ gauge theory

The coupled $2d/4d$ system

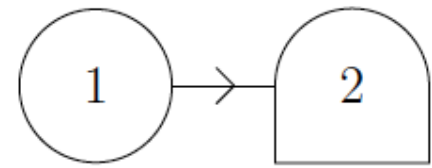


- Understanding the $SU(2)$ node just as a flavour symmetry, we can write the **effective** twisted superpotential W for the CP^1 model as

$$W = 2\pi i t \sigma - 2 \left[\sigma \left(\log \frac{\sigma}{\mu} - 1 \right) \right]$$

(d'Adda+Di Vecchia+ ...'82)

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FI coupling $t = i\xi + \frac{\theta}{2\pi}$

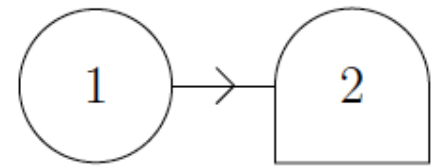
twisted $U(1)$
chiral superfield

UV scale

2 because of the $SU(2)$
flavour symmetry

This is a purely 2d point of view

The coupled $2d/4d$ system



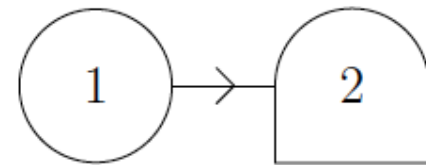
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- Understanding the $SU(2)$ node as a **$4d$ theory**, and giving a vev to the adjoint scalar ϕ , we introduce a twisted mass for σ and W becomes

$$W = 2\pi i t \sigma - \text{Tr} \left[(\sigma - \langle \phi \rangle) \left(\log \frac{\sigma - \langle \phi \rangle}{\mu} - 1 \right) \right] \quad (\text{Hanany+Hori '97})$$

The coupled $2d/4d$ system



- Understanding the SU(2) node just as a flavour symmetry, we can write the **effective** twisted superpotential W for the CP¹ model as

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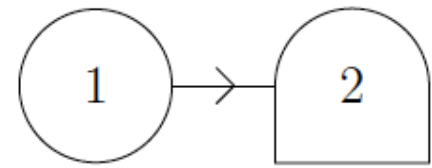
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4d vev's as twisted masses for σ

The **4d** theory is treated only classically

The coupled $2d/4d$ system



- Understanding the $SU(2)$ node just as a flavour symmetry, we can write the **effective** twisted superpotential W for the CP^1 model as

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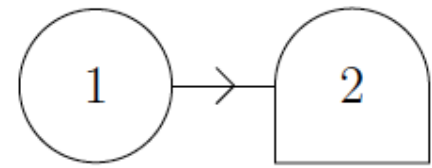
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- Taking into account the **quantum fluctuations of the $4d$ $SU(2)$ theory**, we finally get

(Gaiotto+Gukov+Seiberg '13, ...)

$$W = 2\pi i t \sigma - \left\langle \text{Tr} \left[(\sigma - \phi) \left(\log \frac{\sigma - \phi}{\mu} - 1 \right) \right] \right\rangle$$

The coupled $2d/4d$ system



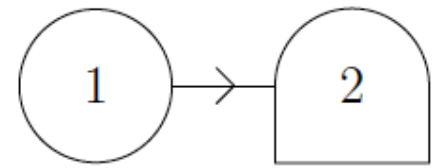
- This $2d/4d$ quiver describes a $4d$ $SU(2)$ theory coupled with a $2d$ $U(1)$ theory inside.
- Thus, it should describe the defect corresponding to

$$SU(2) \rightarrow S[U(1) \times U(1)]$$

and to the partition $\vec{n} = [1, 1]$.

Can we make a quantitative check ?

The coupled $2d/4d$ system



- This $2d/4d$ quiver describes a $4d$ $SU(2)$ theory coupled with a $2d$ $U(1)$ theory inside.
- Thus, it should describe the defect corresponding to

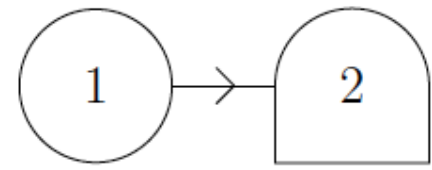
$$SU(2) \rightarrow S[U(1) \times U(1)]$$

and to the partition $\vec{n} = [1, 1]$.

Can we make a quantitative check ?

YES !

The coupled $2d/4d$ system



- Consider the twisted superpotential

$$W = 2\pi i t \sigma - \left\langle \text{Tr} \left[(\sigma - \phi) \left(\log \frac{\sigma - \phi}{\mu} - 1 \right) \right] \right\rangle$$

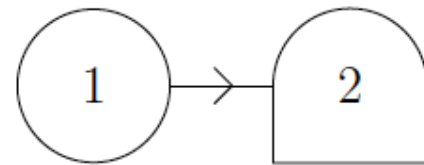
$$= - \left\langle \text{Tr} \left[(\sigma - \phi) \left(\log \frac{\sigma - \phi}{\Lambda_1} - 1 \right) \right] \right\rangle$$

- The $2d$ vacuum is determined by

$$\exp \left(\frac{\partial W}{\partial \sigma} \right) = 1 \quad \Longleftrightarrow \quad \exp \left\langle \text{Tr} \left[\log \frac{\sigma - \phi}{\Lambda_1} \right] \right\rangle = 1$$

$2d$ dynamically generated scale $\Lambda_1^2 = e^{2\pi i t} \mu^2$

The coupled $2d/4d$ system



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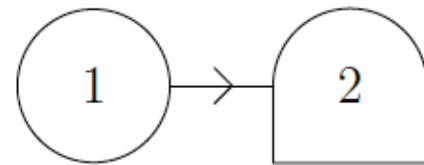
$$\exp \left(\frac{\partial W}{\partial \sigma} \right) = 1 \iff \exp \left\langle \text{Tr} \left[\log \frac{\sigma - \phi}{\Lambda_1} \right] \right\rangle = 1$$

- This equation can be solved using the Seiberg-Witten curve of the $4d$ $SU(2)$ theory

$$y^2 = P_2(x)^2 - 4\Lambda^4$$

$4d$ dynamically generated scale $\Lambda^4 = e^{2\pi i \tau} \mu^4$
 characteristic polynomial of SW $P_2(x) = x^2 - \frac{1}{2} \langle \text{Tr} \phi^2 \rangle$

The coupled $2d/4d$ system



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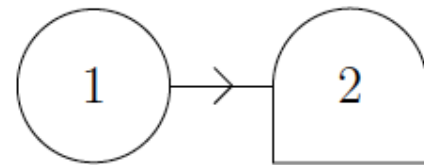
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$$P_2(x) = x^2 - \frac{1}{2} \langle \text{Tr} \phi^2 \rangle$$

The coupled $2d/4d$ system



- From the Seiberg-Witten relation $\left\langle \text{Tr} \frac{1}{x - \phi} \right\rangle = \frac{P_2'(x)}{y}$, one gets

$$\begin{aligned} \left\langle \text{Tr} \left[\log \frac{\sigma - \phi}{\Lambda_1} \right] \right\rangle &= \int d\sigma \left\langle \text{Tr} \left[\frac{1}{\sigma - \phi} \right] \right\rangle = \dots \\ &= \log \left(\frac{P_2(\sigma) + \sqrt{P_2(\sigma)^2 - 4\Lambda^4}}{2\Lambda_1^2} \right) \end{aligned}$$

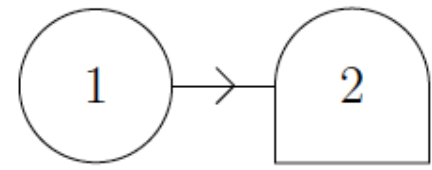
- With simple manipulations, the **vacuum equation** becomes

$$\exp \left(\frac{\partial W}{\partial \sigma} \right) = 1 \quad \Longleftrightarrow \quad P_2(\sigma) = \Lambda_1^2 + \frac{\Lambda^4}{\Lambda_1^2}$$

which is solved by

$$\sigma_\star = \sqrt{\frac{1}{2} \langle \text{Tr} \phi^2 \rangle + \Lambda_1^2 + \frac{\Lambda^4}{\Lambda_1^2}} = \underbrace{a}_{\text{v.e.v. of } \phi} + \frac{1}{2a} \left(\underbrace{\Lambda_1^2}_{\text{instanton terms}} + \frac{\Lambda^4}{\Lambda_1^2} \right) + \dots$$

The coupled $2d/4d$ system



- Evaluating W on the vacuum solution σ_* , we find

$$W(\sigma_*)|_{\text{inst}} = \frac{1}{2a} \left(\Lambda_1^2 - \frac{\Lambda^4}{\Lambda_1^2} \right) + \dots$$

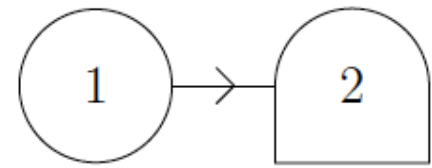
- Notice that

$$\Lambda_1^2 \sim e^{2\pi i t} \rightarrow q_1 \quad , \quad \frac{\Lambda^4}{\Lambda_1^2} \sim e^{2\pi i \tau} e^{-2\pi i t} \rightarrow \frac{q}{q_1} = q_2$$

and thus

$$W(\sigma_*)|_{\text{inst}} = \frac{q_1}{2a} - \frac{q_2}{2a} + \dots$$

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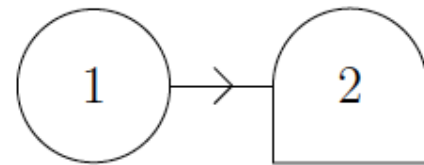
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- Furthermore, we see that $\Lambda_1 \gg \Lambda$

The coupled $2d/4d$ system



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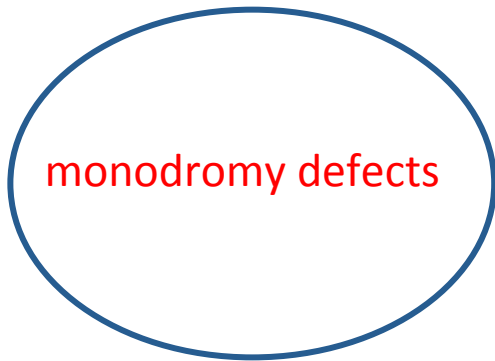
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$$W(\sigma_*)|_{\text{inst}} = \frac{q_1}{2a} - \frac{q_2}{2a} + \dots$$

This exactly matches the twisted superpotential due to ramified instantons !!

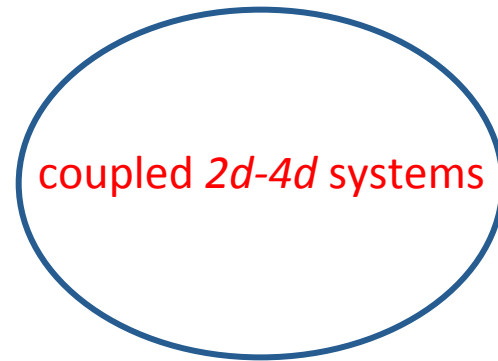
Recap



$4d \longrightarrow 2d$

W_{inst} from localization

ramified instanton
counting parameters q_i



$2d \longrightarrow 4d$

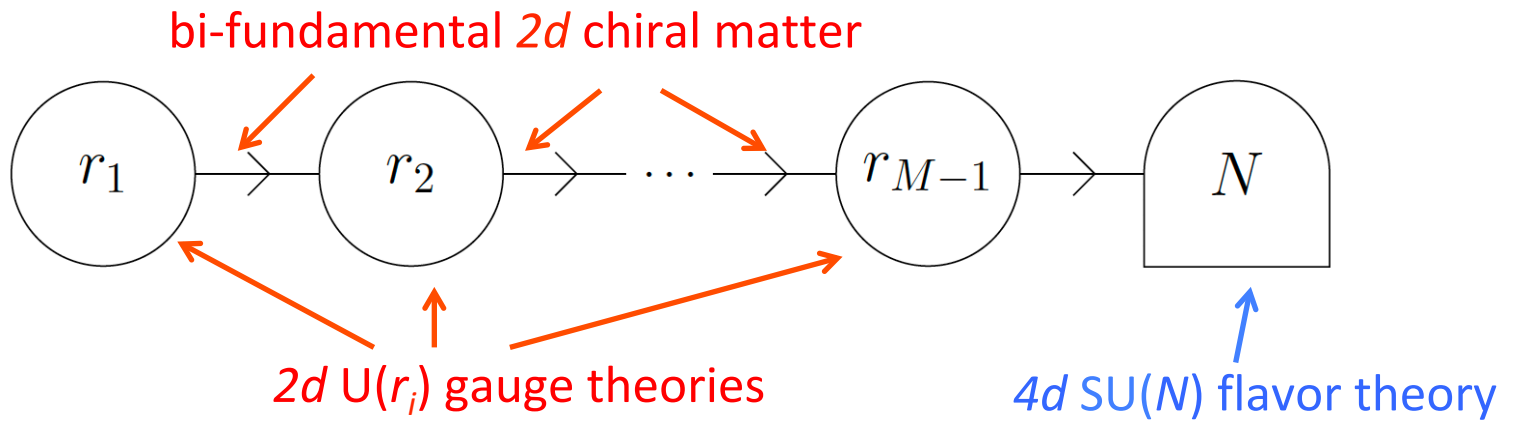
W_{inst} from vacuum equations

dynamically generated
scales Λ_i

$$W_{\text{inst}} \Big|_{\text{loc}} = W(\sigma) \Big|_{\text{vacuum}}$$

General case

- The quiver which describes the general defect in the $SU(N)$ theory is



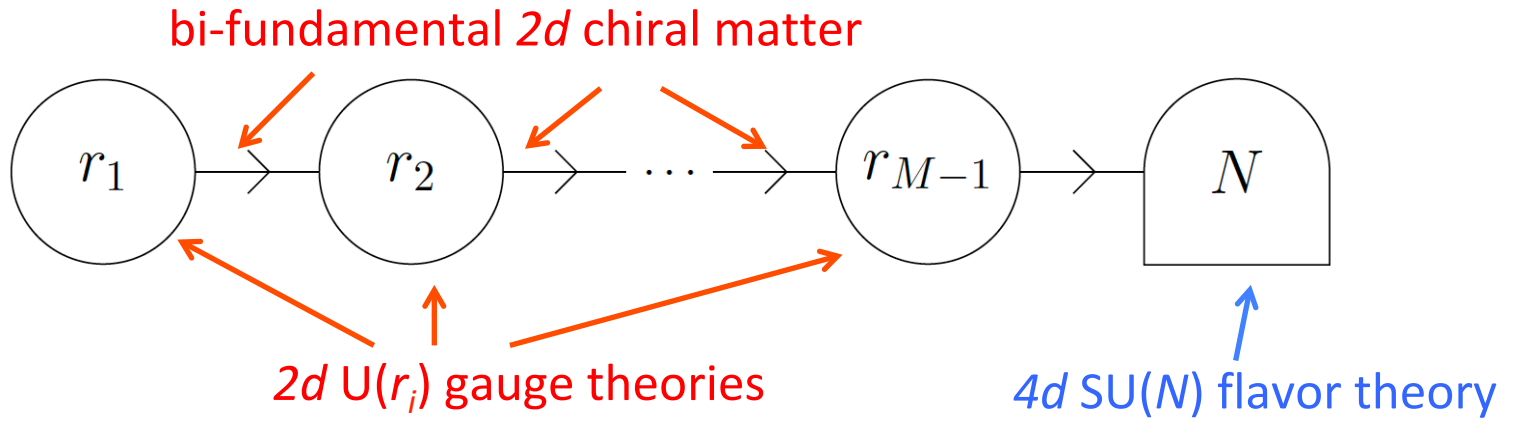
- This quiver corresponds to the monodromy defect with

$$\vec{n} = [n_1, \dots, n_M] \quad r_i = \sum_{j=1}^i n_j$$

$$SU(N) \rightarrow S[U(n_1) \times U(n_2) \times \dots \times U(n_M)]$$

General case

- The quiver which describes the general defect in the $SU(N)$ theory is



- Given the structure of the quiver, the twisted superpotential is

$$\begin{aligned}
 W = 2\pi i & \sum_{i=1}^{M-1} \sum_{s=1}^{r_i} t_i \sigma_s^{(i)} - \sum_{i=1}^{M-2} \sum_{s=1}^{r_i} \sum_{t=1}^{r_{i+1}} \varpi(\sigma_s^{(i)} - \sigma_t^{(i+1)}) \\
 & - \sum_{s=1}^{r_{M-1}} \left\langle \text{Tr } \varpi(\sigma_s^{(M-1)} - \phi) \right\rangle
 \end{aligned}$$

$$\varpi(x) = x \log \left(\frac{x}{\mu} - 1 \right)$$

General case

- The non-perturbative contributions to W are computed from the vacuum equations

$$\exp \left(\frac{\partial W}{\partial \sigma_s^{(i)}} \right) = 1$$

and **exactly match** those obtained from the localization approach!

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Technical but important point

- The ramified instanton partition function is given in terms of an multiple integral; for example at 1-instanton we have

$$Z_1 = - \sum_{i=1}^M \frac{q_i}{\epsilon_1} \int \frac{d\chi_i}{2\pi i} \prod_{s=1}^{n_i} \frac{1}{\left(a_s - \chi_i + \frac{\epsilon_1 + \epsilon_2}{2} \right)} \prod_{t=1}^{n_{i+1}} \frac{1}{\left(\chi_i - a_t + \frac{\epsilon_1 + \epsilon_2}{2} \right)}$$

- Thus we have to specify the **contour of integration**, or equivalently specify **which poles contribute to the integral**

Integration contour

- The precise match is obtained by giving a positive imaginary part to the ϵ parameters and selecting the contribution from the following poles:

$$\chi_i = a_s + \frac{\epsilon_1 + \epsilon_2}{2} \quad s = 1 \cdots, n_i, \quad i = 1, \cdots, M - 1$$

$$\chi_M = a_t - \frac{\epsilon_1 + \epsilon_2}{2} \quad t = 1 \cdots, n_1$$

(Gorsky+LeFloch+... '17
Ashok+Billo+... '17,' 18)

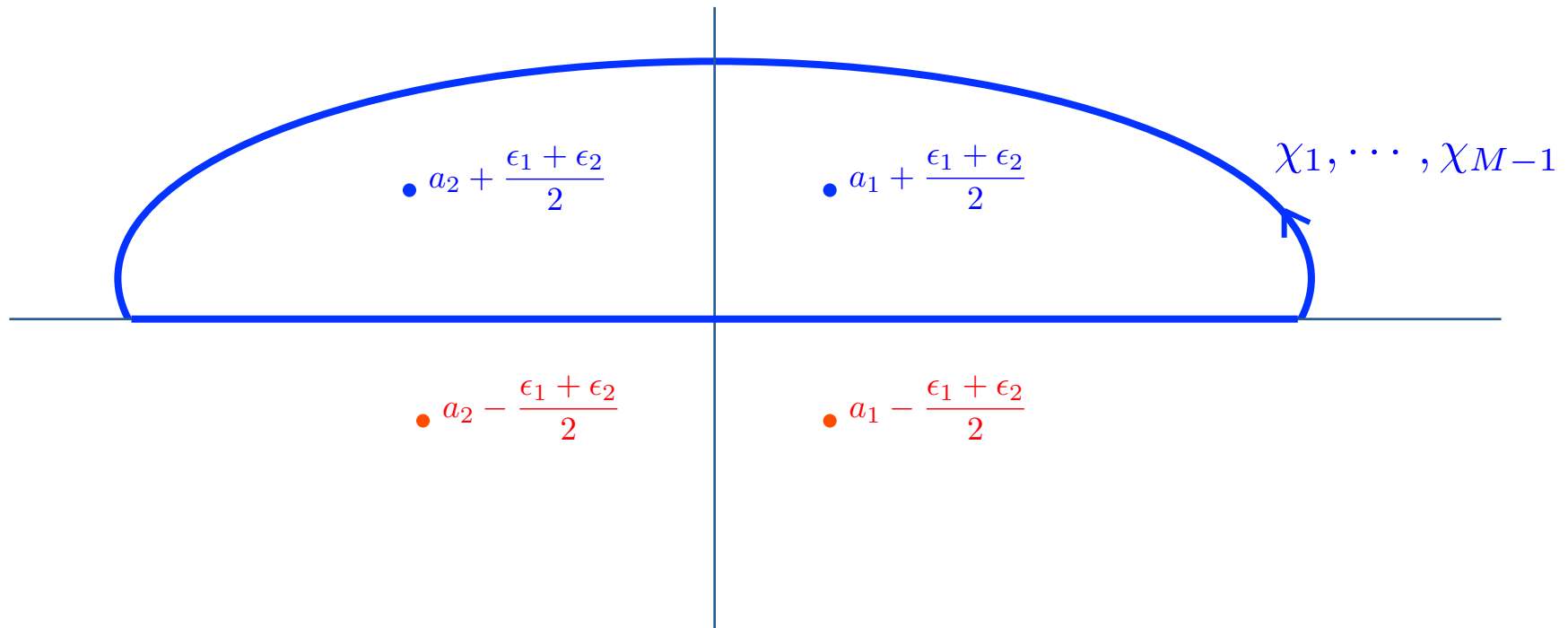
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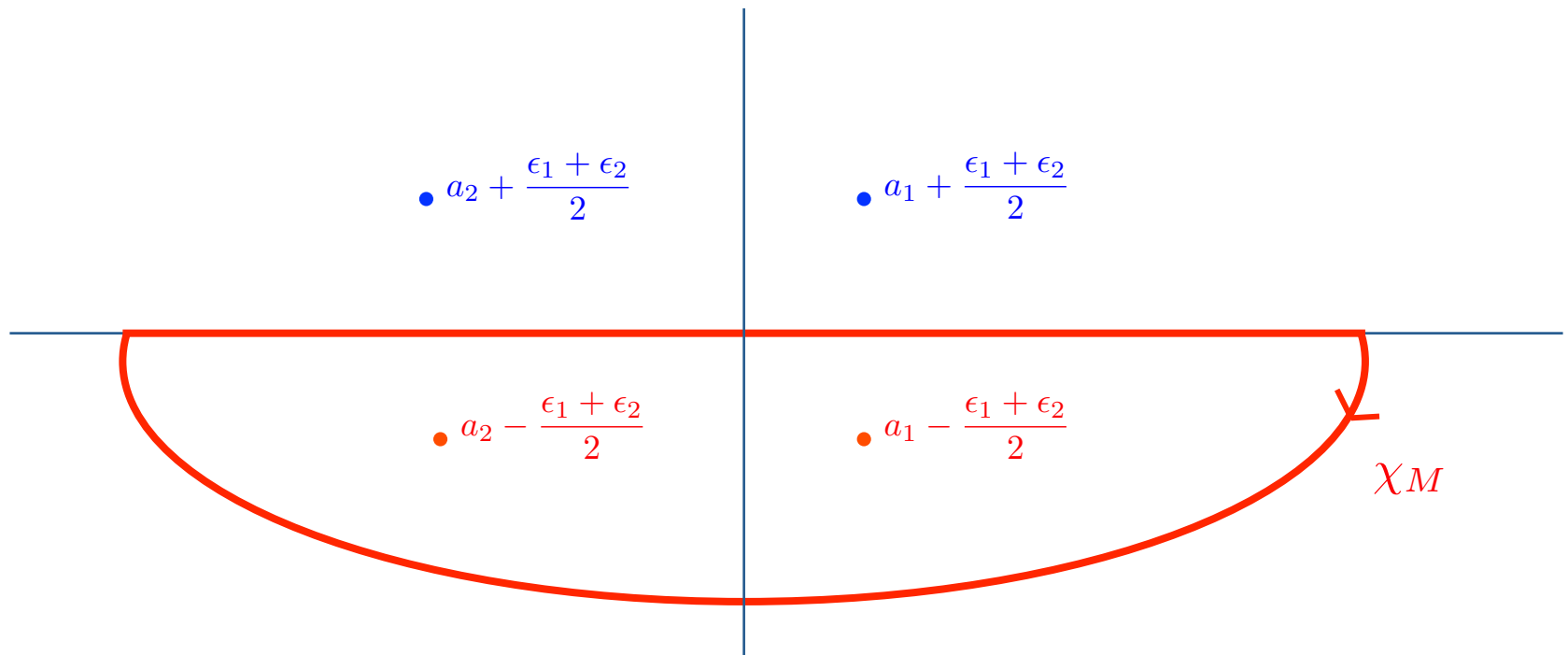
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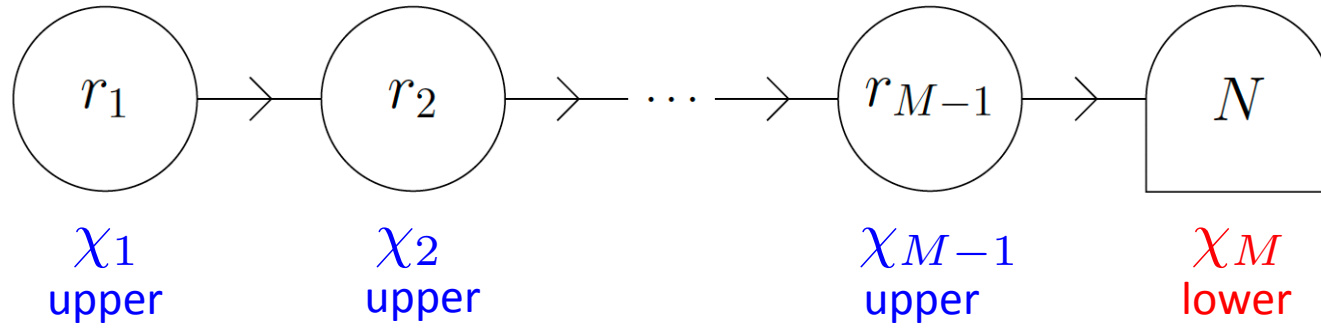
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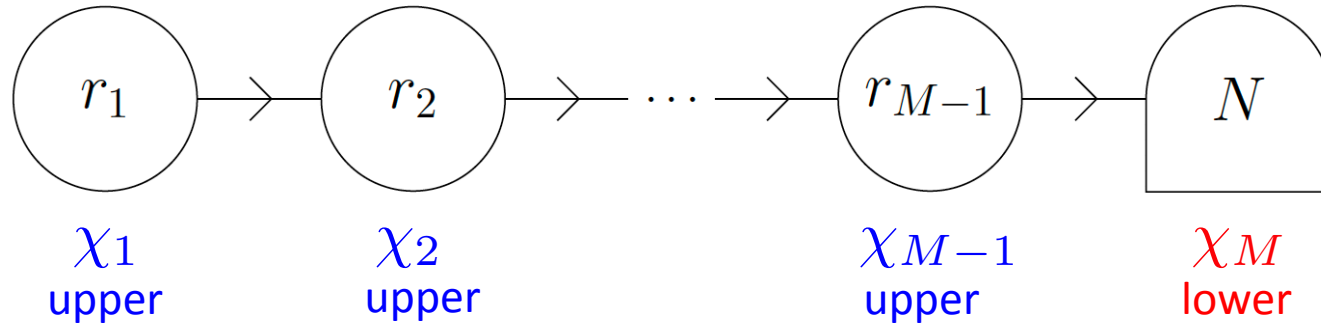
(Gorsky+LeFloch+... '17
Ashok+Billo+... '17,' 18)



Integration contour



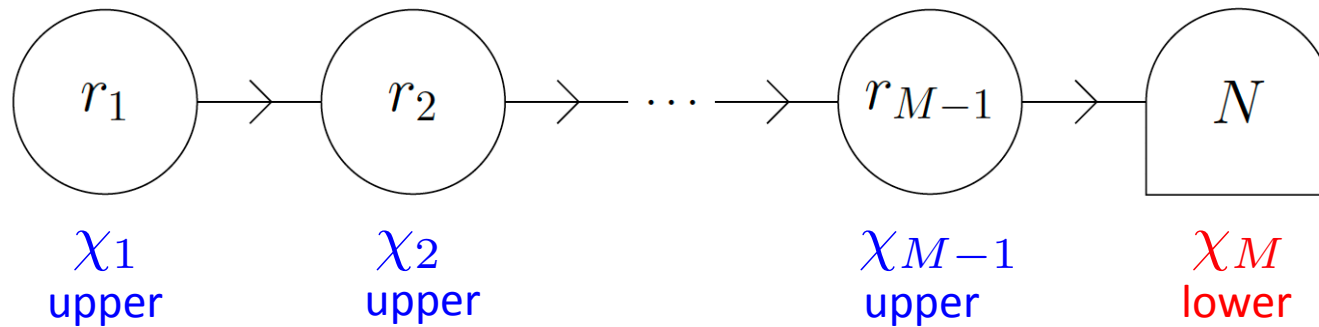
Integration contour



- This integration prescription can be elegantly specified with a **Jeffrey-Kirwan vector η** given by

$$\eta = -\chi_1 - \chi_2 \cdots - \chi_{M-1} + \xi_M \chi_M \quad (\xi_M \gg 1)$$

Integration contour

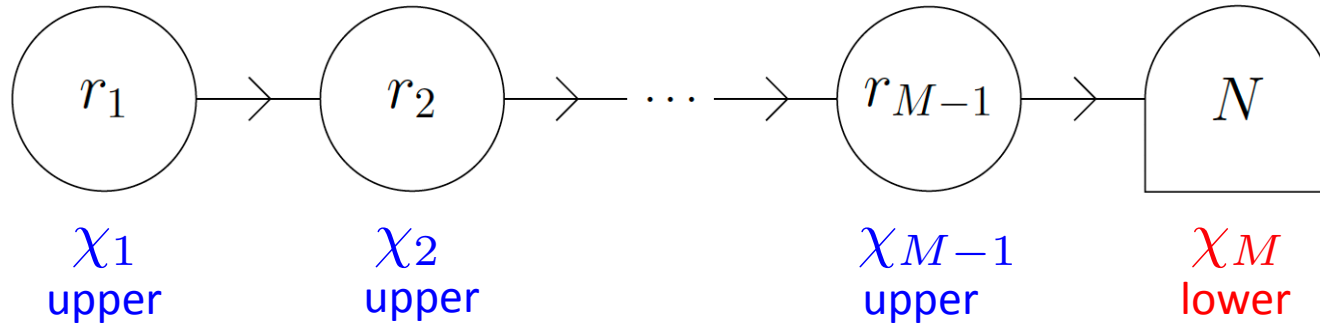


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- sign of the beta-function for the FI couplings of the $U(r_i)$

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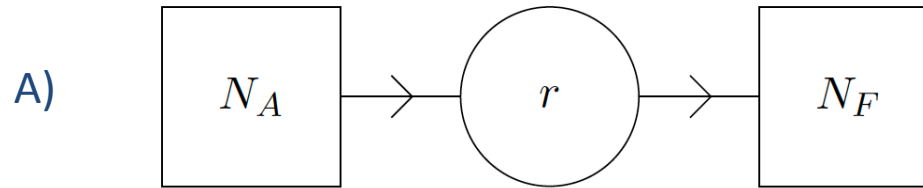
FI couplings of the $U(r_i)$ nodes of the quiver

$$\xi_1 \ll \xi_2 \ll \cdots \ll \xi_{M-1} \ll \xi_M \Leftrightarrow \Lambda_1 \gg \Lambda_2 \gg \cdots \gg \Lambda_{M-1} \gg \Lambda$$

Duality

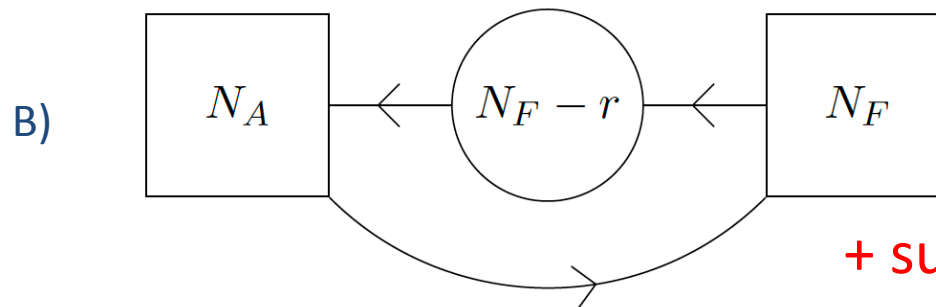
Duality

- Consider a $2d$ $U(r)$ theory with N_F fundamentals and N_A ($< N_F$) anti-fundamentals



- This theory is dual to a $2d$ $U(N_F - r)$ theory with the roles of fundamentals and anti-fundamentals reversed

(Seiberg '94)

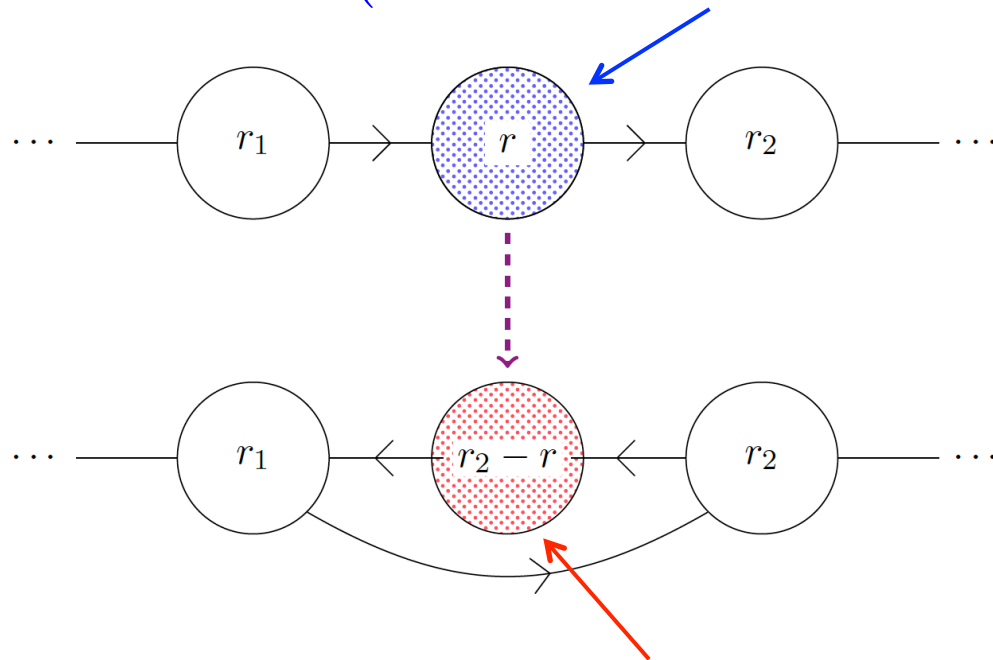


$$W_{\text{class}}^A = 2\pi i t \text{Tr} \sigma \quad \rightarrow \quad W_{\text{class}}^B = -2\pi i t \text{Tr} \sigma + 2\pi i t \sum_{f=1}^{N_F} m_f$$

Duality

- We can apply this duality rule to the quiver representing a surface defect

$$W_{\text{class}} = \cdots + 2\pi i (t_1 \text{Tr } \sigma^1 + t \text{Tr } \sigma + t_2 \text{Tr } \sigma^2) + \cdots$$

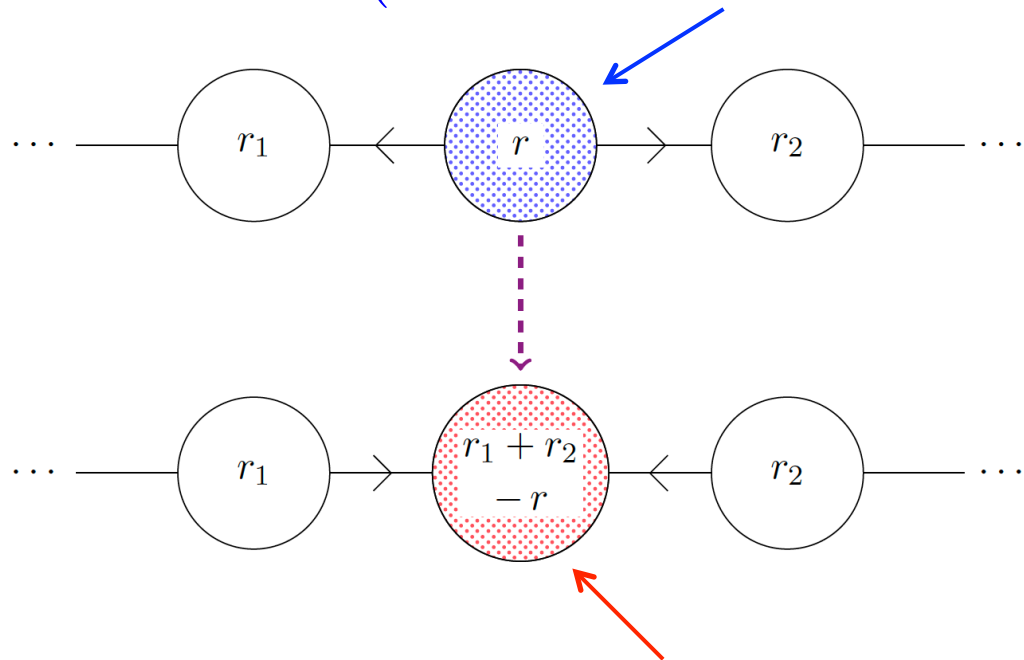


$$W_{\text{class}} = \cdots + 2\pi i (t_1 \text{Tr } \sigma^1 - t \text{Tr } \sigma + (t_2 + t) \text{Tr } \sigma^2) + \cdots$$

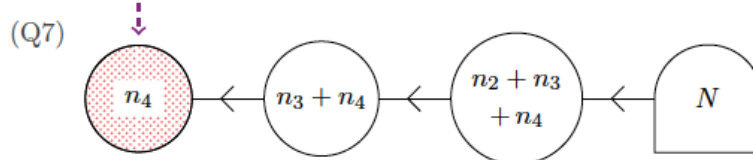
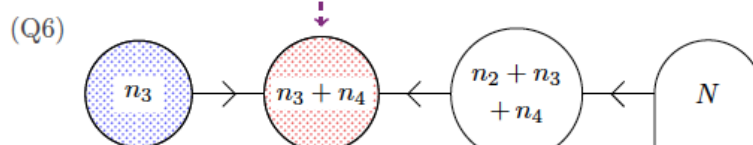
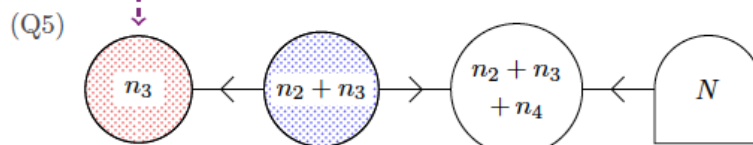
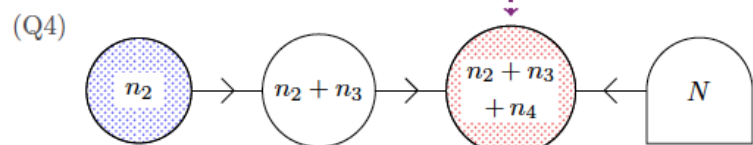
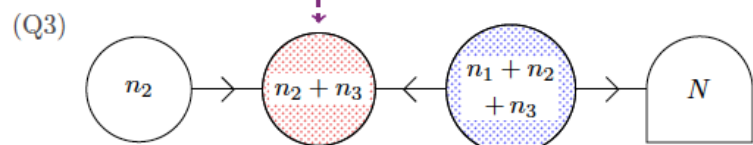
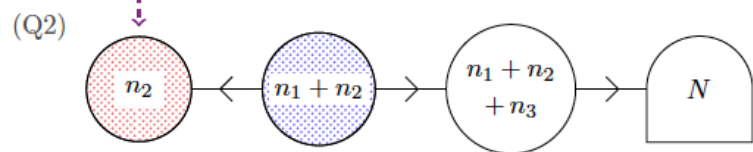
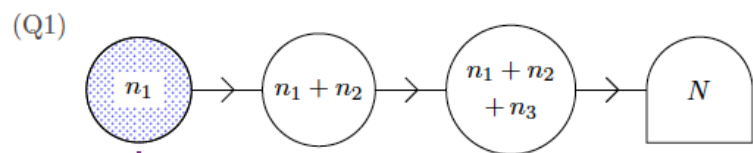
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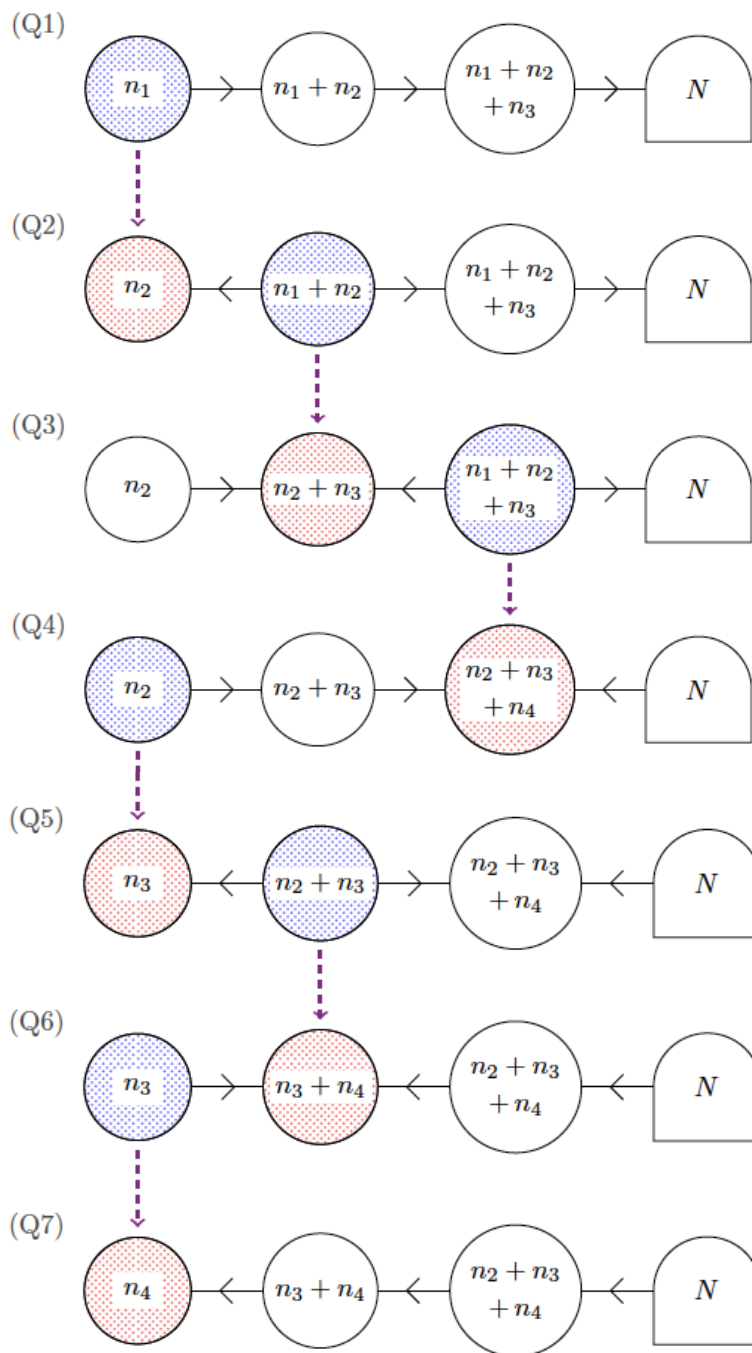
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$$W_{\text{class}} = \cdots + 2\pi i ((t_1 + t) \text{Tr } \sigma^1 - t \text{Tr } \sigma + (t_2 + t) \text{Tr } \sigma^2) + \cdots$$



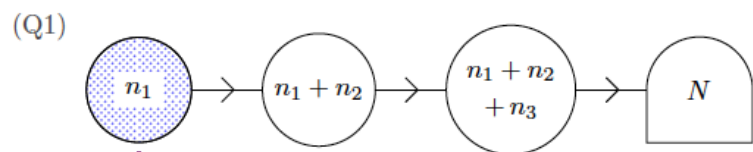


All these different quiver theories provide alternative descriptions of the **same surface operator** corresponding to

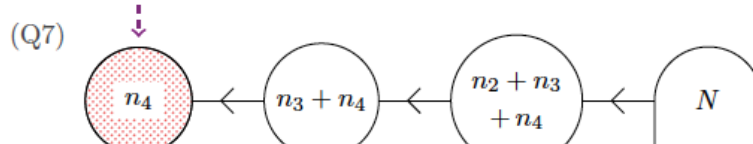
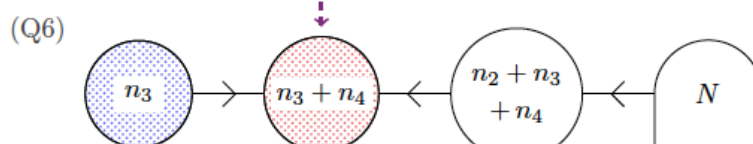
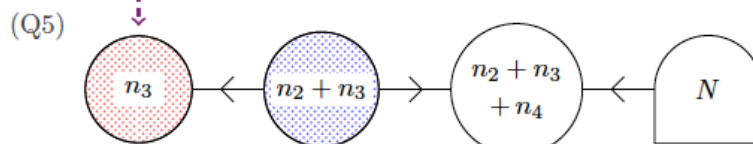
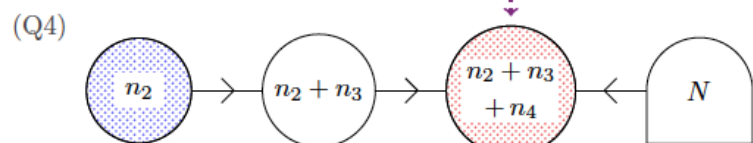
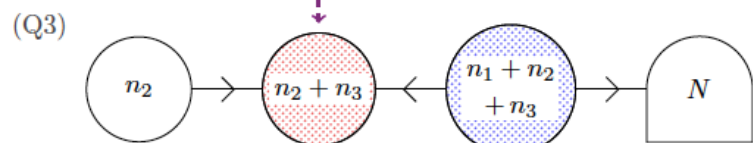
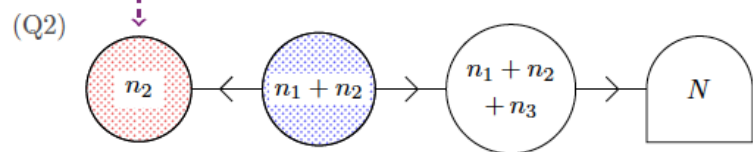
$$\mathrm{SU}(N)$$

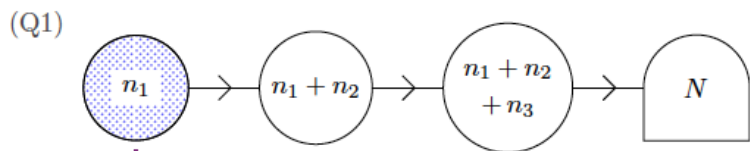


$$\mathrm{S}[\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{U}(n_3) \times \mathrm{U}(n_4)]$$

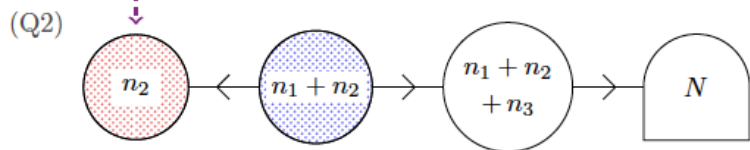


$$W_{\text{class}} = 2\pi i(t_1 \text{Tr } \sigma^1 + t_2 \text{Tr } \sigma^2 + t_3 \text{Tr } \sigma^3)$$

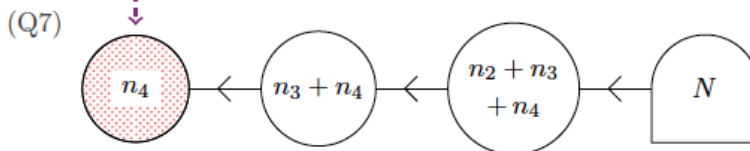
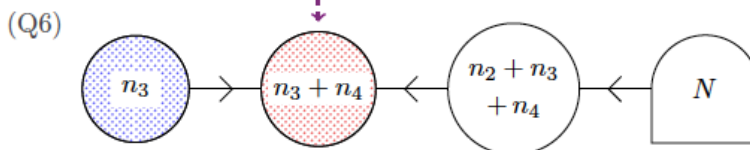
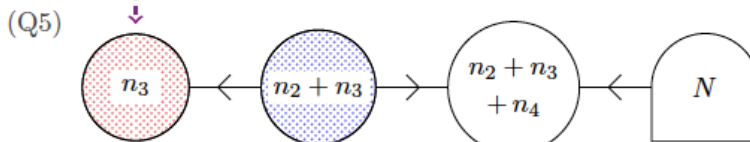
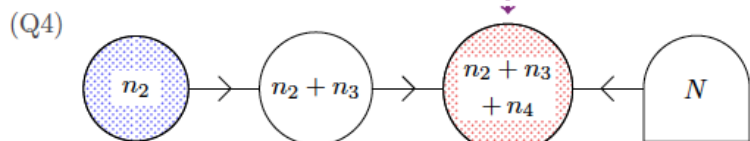
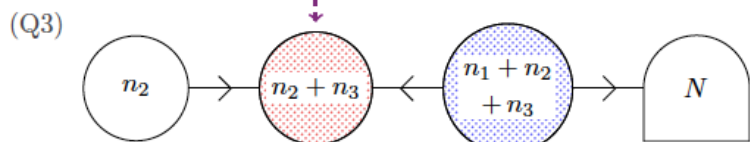


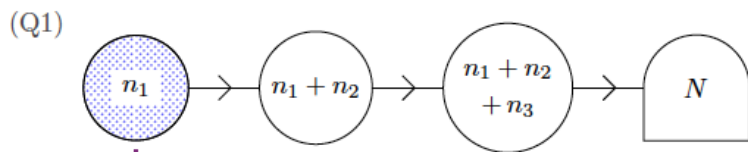


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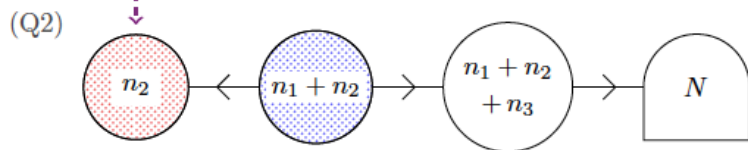


$$W_{\text{class}} = 2\pi i(-t_1 \text{Tr } \sigma^1 + (t_1 + t_2) \text{Tr } \sigma^2 + t_3 \text{Tr } \sigma^3)$$

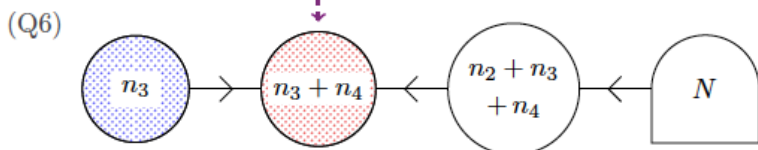
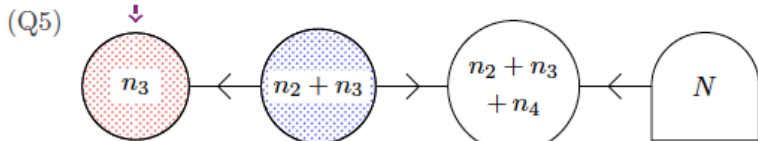
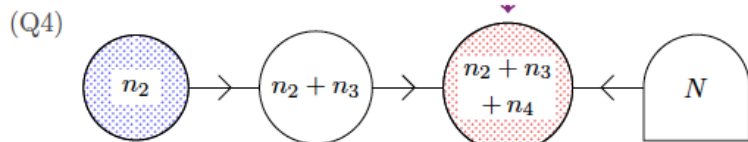
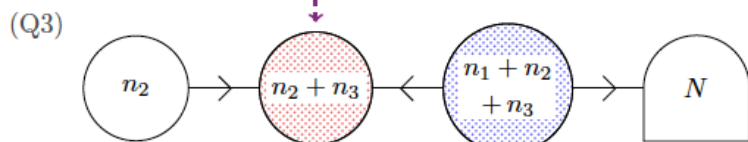




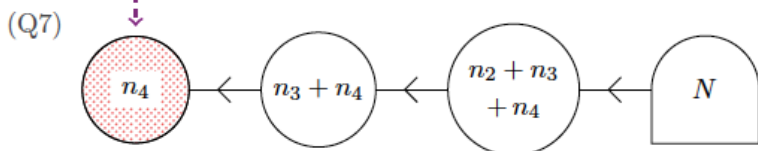
$$W_{\text{class}} = 2\pi i(t_1 \text{Tr } \sigma^1 + t_2 \text{Tr } \sigma^2 + t_3 \text{Tr } \sigma^3)$$



$$W_{\text{class}} = 2\pi i(-t_1 \text{Tr } \sigma^1 + (t_1 + t_2) \text{Tr } \sigma^2 + t_3 \text{Tr } \sigma^3)$$



$$W_{\text{class}} = 2\pi i(t_3 \text{Tr } \sigma^1 - (t_2 + t_3) \text{Tr } \sigma^2 - t_1 \text{Tr } \sigma^3)$$

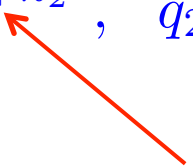


$$W_{\text{class}} = 2\pi i(-t_3 \text{Tr } \sigma^1 - t_2 \text{Tr } \sigma^2 - t_1 \text{Tr } \sigma^3)$$

⋮

Duality

- For any of these quiver theories we can work out the q vs Λ map just as we did for the simplest quiver
- For example, after one duality step we find

$$q_1 \sim \Lambda_1^{n_1+n_2}, \quad q_2 \sim \frac{\Lambda_2^{n_1+2n_2+n_3}}{\Lambda_1^{n_1+n_2}}, \quad q_3 \sim \Lambda_3^{n_3+n_4}, \quad q_4 \sim \frac{\Lambda^{2N}}{q_1 q_2 q_3}$$


Note that the exponents are the (absolute values of the) beta-functions for the FI couplings ξ_i , and $2N$ is the beta-function of the $4d$ $SU(N)$ theory.

Duality

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- Moreover, the sign change in the FI term of the first node

$$W_{\text{class}} = 2\pi i \left(-t_1 \text{Tr } \sigma^1 + (t_1 + t_2) \text{Tr } \sigma^2 + t_3 \text{Tr } \sigma^3 \right)$$

leads to consider a new Jeffrey-Kirwan vector

$$\eta = +\xi_1 \chi_1 - \xi_2 \chi_2 \cdots - \xi_{M-1} \chi_{M-1} + \xi_M \chi_M$$

 χ_1 is now integrated in the lower plane !

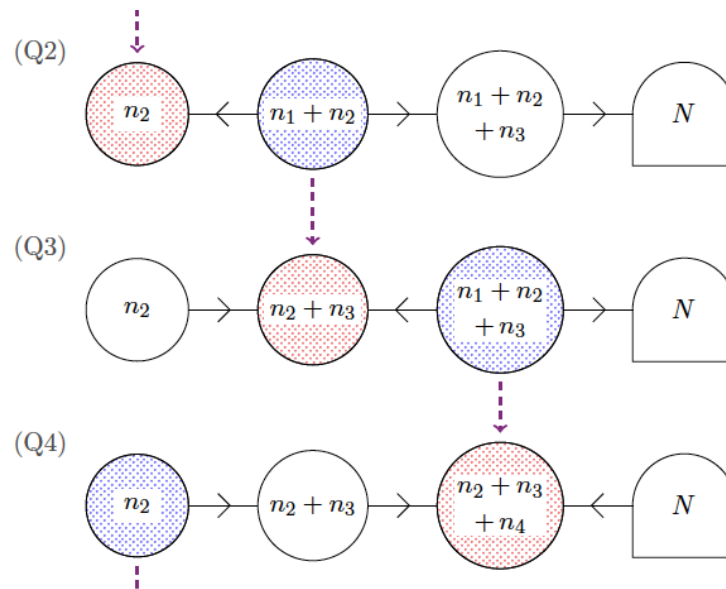
Duality

- Solving the vacuum equations

$$\exp\left(\frac{\partial W}{\partial \sigma}\right) = 1$$

leads again to **a perfect match** (term by term) with the **superpotential of ramified instantons with the new JK parameter!**

- The same is true for any other quiver diagram of the sequence we considered



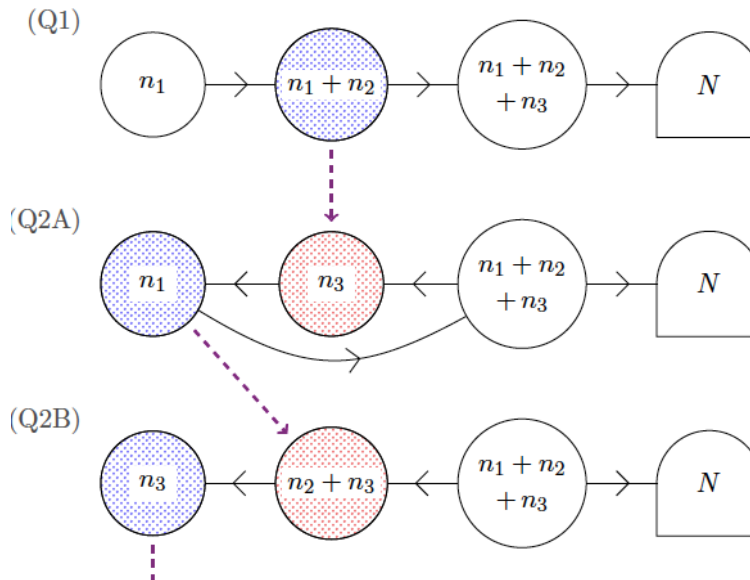
Duality

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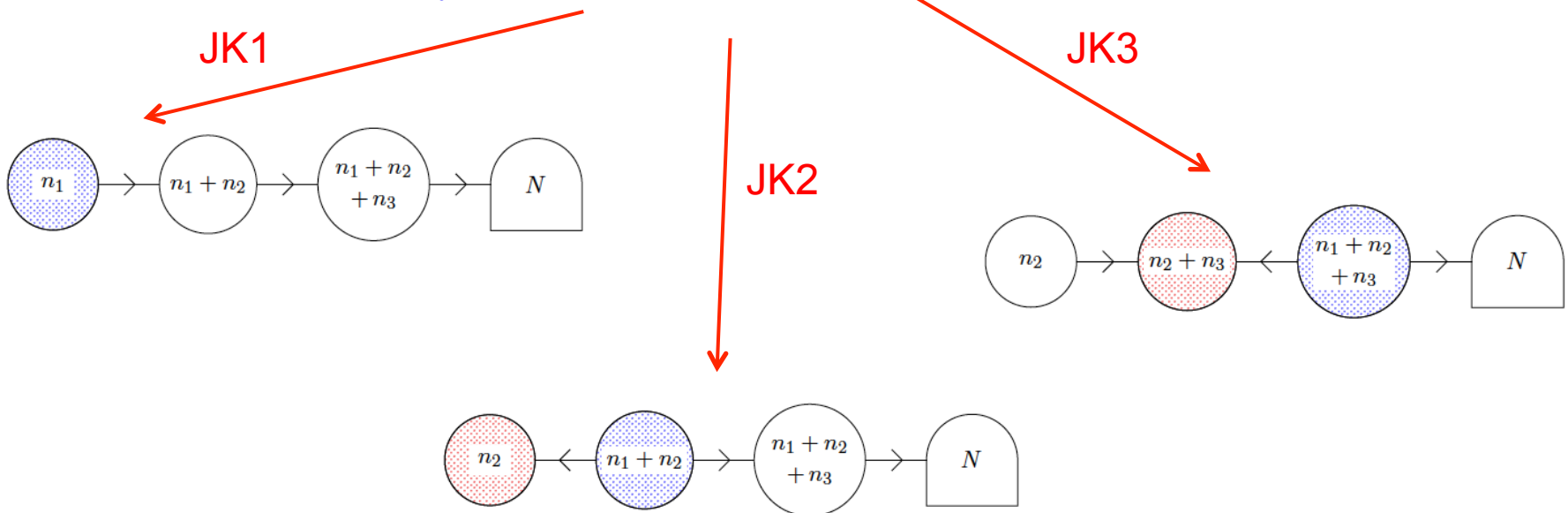
- The same is true for any other quiver diagram of the sequence we considered, and also in **sequences with non-linear quivers**



Duality

- Starting from the **same localization integrand** and choosing **different integration contours** (or **different Jeffrey-Kirwan vectors**) we obtain a perfect match with **different quiver theories** related to each other **by duality transformations**

$$\prod_{i=1}^M \oint \frac{d\chi_i}{2\pi i} \frac{q_i^{d_i}}{d_i!} Z_{d_i}(a, \epsilon_1, \epsilon_2)$$



Conclusions

Summary of results

monodromy defects

- $[n_1, n_2, \dots, n_M]$
- $4d$ v.e.v's of $SU(N)$
- Ramified instanton counting parameters q_i
- W_{inst} from localization
- Integration contour

coupled $2d$ - $4d$ systems

- Ranks of gauge groups
- $2d$ twisted masses
- Dynamically generates scales Λ_i
- W_{inst} from vacuum eqs.
- Duality frame

Conclusions

- Similar analysis can be done for other type of theories
 - $N=2^*$ theories
 - SQCD's (with matter in fundamental representation)
 - Chern-Simons like theories in $3d/5d$
 - Theories with other groups (orthogonal, symplectic,...)
- Explore the connection with integrable models
- ...

**Thanks a lot for your
attention!**