The Moduli Space of Instantons from 3d $\mathcal{N}=3$ and $\mathcal{N}=4$ Gauge Theories

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Based on: work in progress with Cremonesi & Zaffaroni
1508.0613 by N. M.
1408.6835 with Cremonesi, Ferlito & Hanany
1403.2384 with Cremonesi, Hanany & Zaffaroni
1309.2657 by Cremonesi, Hanany & Zaffaroni

PART I

3d $\mathcal{N}=4$ gauge theories
and
the moduli space of instantons on $\mathbb{C}^2$

3d $\mathcal{N}=4$ gauge theories

- Two branches of the moduli space
  - both are hyperKähler cones.

1. Higgs branch: parametrised by VEVs of the hypermultiplets
   - Classically exact
   - Suitable description (in $\mathcal{N}=2$ language): Gauge invariant combinations of the chiral fields from the hypers subject to the F-terms.

2. Coulomb branch: parametrised by VEVs of the vector multiplets
   - Quantum corrections
   - Suitable description: Monopole operators dressed by the Casimirs of adjoint scalars in the v-plet.

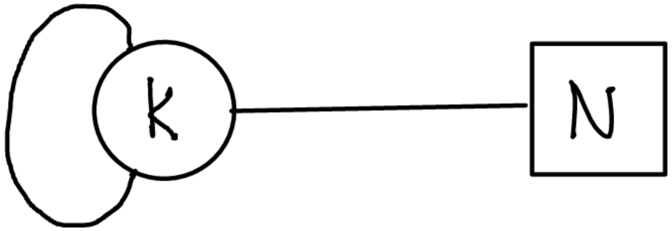
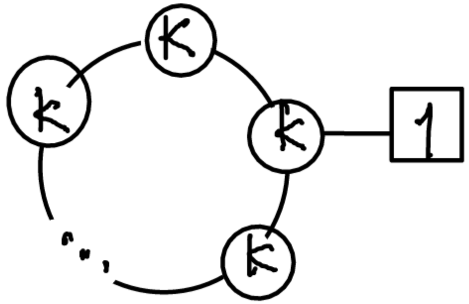
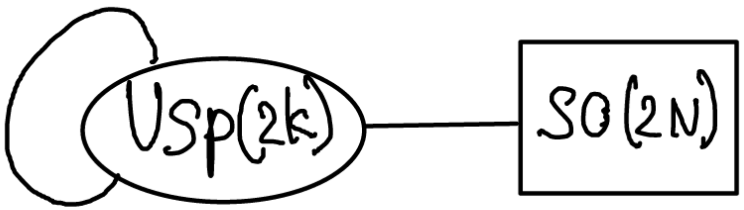
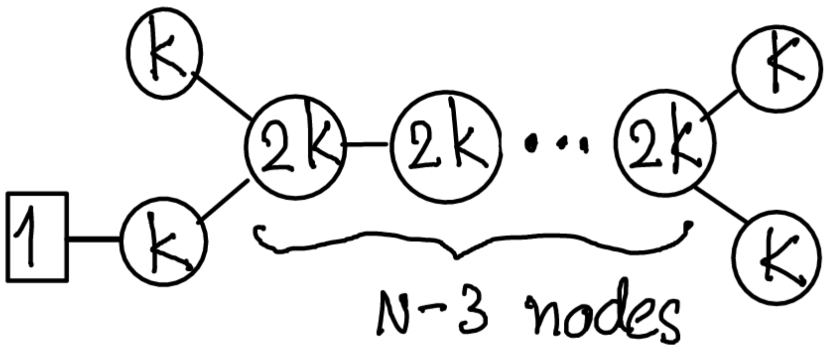
- Mirror symmetry: If theories A and B are mirror dual to each other, the Higgs (Coulomb) branch of A is equal to the Coulomb (Higgs) branch of B. [Intriligator, Seiberg '96]

- The Higgs and Coulomb branches of certain 3d $\mathcal{N}=4$ theories are isomorphic to the moduli spaces of YM instantons on flat and ALE spaces

[Atiyah, Drinfeld, Hitchin, Manin '78; Kronheimer Nakajima '90
Douglas '95, Witten '95 ; Intriligator '97, Witten '09
Dey, Hanany, N.M., Rodriguez-Gomez, Seong '09

Intriligator, Seiberg '96 ; de Boer, Hori, Ooguri, Oz '96
Cremonesi, Ferlito, Hanany, NM '14 ; Porrati, Zaffaroni '96
Braverman, Finkelberg, Nakajima '16 ; N.M. '15]

The moduli space of k G-instantons on $\mathbb{C}^2$

G	Higgs of (ADHM quiver)	Caulomb of
SU(N)		 (N circular nodes)
SO(2N)	A 	

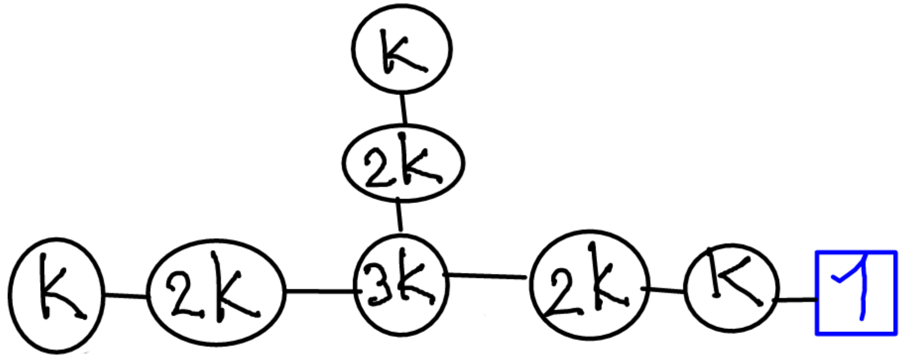
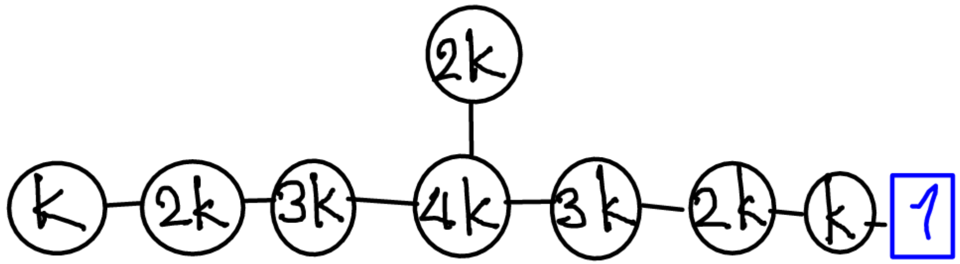
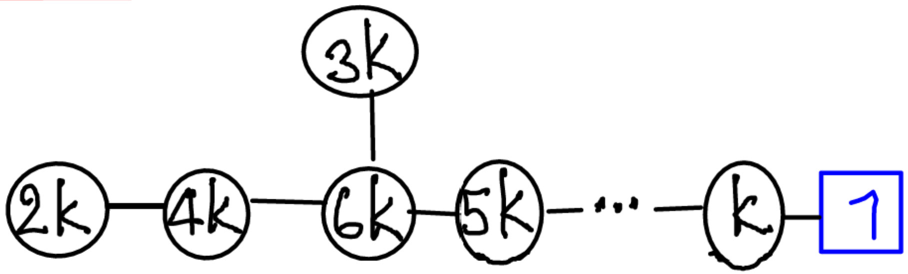
Comments:

1. The quivers in the rightmost column are just affine Dynkin diagram of G with
   (i) gauge groups being $U(ka_i^\vee)$; $\left[a_i^\vee = \text{dual coxeter label of the } i^{th}\text{-node} \right]$
   (ii) $U(1)$ flavour node attached to the affine node.

2. For each G, the pair of theories are mirror duals.

3. The ADHM quiver can be realised using the system of D2-D6 branes (possibly with an orientifold plane).
   The corresponding mirror theory can be obtained by applying T-duality and then S-duality on the system. [Hanany-Witten '96]

The moduli space of k G-instantons on $\mathbb{C}^2$

G	Higgs of theories on M5-branes on $S^1 \times$ Riemann surface with punctures	Coulomb of
E_6	$(k^3), (k^3), (k^2, k-1, 1)$	
E_7	$(k^4), ((2k)^2), (k^3, k-1, 1)$	
E_8	$((3k)^2), ((2k)^3), (k^5, k-1, 1)$	

Comments:

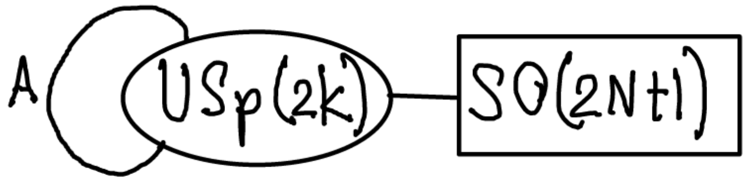
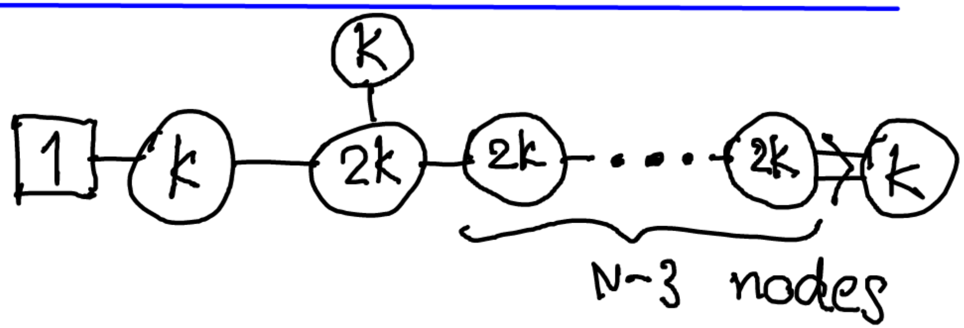
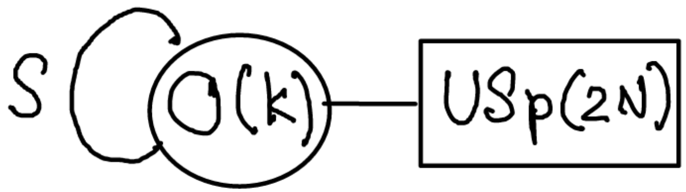
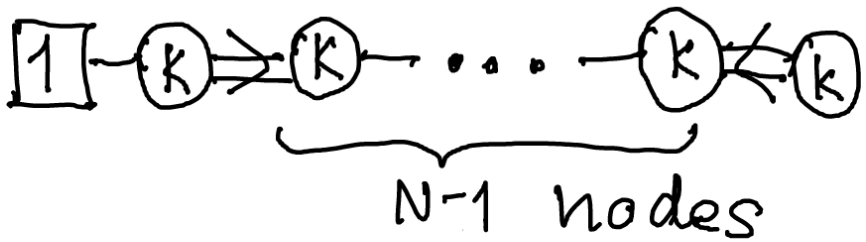
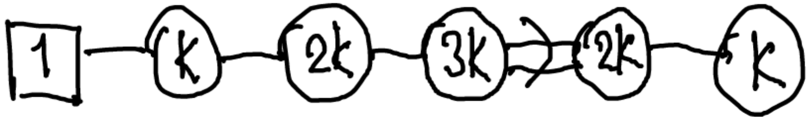
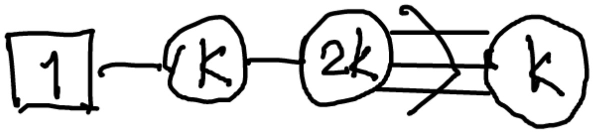
- The analogs of the ADHM quivers are theories on M5-branes wrapping $S^1 \times$ Riemann surface with appropriate punctures. But these have no known Lagrangians.

  [Gaiotto '09; Benini, Benvenuti, Tachikawa '09; Gaiotto, Razamat '12]

- However the mirror theories have Lagrangian descriptions — they are affine $E_{6,7,8}$ Dynkin diagrams with the $U(1)$ flavour node attached at the affine root.

  [Benini, Tachikawa, Xie '10]

The moduli space of k G-instantons on $\mathbb{C}^2$

G	Higgs of	Coulomb of
$SO(2N+1)$		
$USp(2N)$		
G_2		
F_4		

Comments: [Cremonesi, Ferlito, Hanany, NM '14]

- The double and triple laces are not "bifundamental" hypers; there is no known Lagrangian description associated with them.

- Nevertheless, there is a prescription that allows us to study the Coulomb branch of such theories as an algebraic variety.

$\longrightarrow$ The generating function (Hilbert series) of the Coulomb branch operators can be computed

$\equiv$ The instanton partition function at any instanton number.

[Nekrasov '02; Nakajima, Yoshioka '03, '05; Keller, N.M., Song, Tachikawa '11]

PART II

The moduli space of instantons on $\mathbb{C}^2/\Gamma$

$(\Gamma = \hat{A}_{n-1}, \hat{D}_{n+2}, \hat{E}_{6,7,8})$

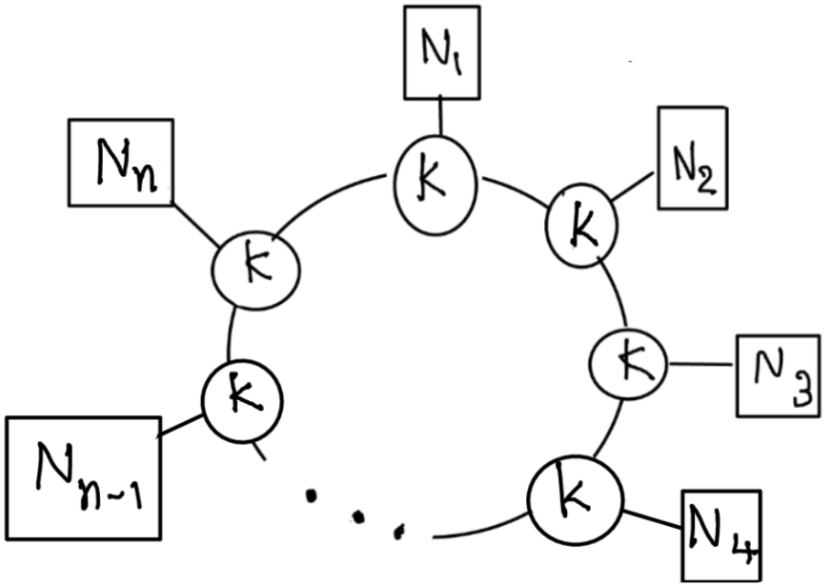
SU(N) instantons on $\mathbb{C}^2/\mathbb{Z}_n$

- In addition to the gauge group SU(N) and the orbifold $\mathbb{C}^2/\mathbb{Z}_n$, we need to specify
  1) the holonomy of the gauge field @ infinity,
  2) the holonomy of the gauge field @ the origin.
- This is to specify how the SU(N) gauge group is broken into its subgroups @ infinity / the origin.
- For simplicity, we consider only the configurations such that 1) = 2).

- [Kronheimer, Nakajima '90] tells us that the moduli space of k $SU(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$ such that

$$SU(N) \to \left(U(N_1)\times U(N_2)\times \dots \times U(N_n)\right)/U(1), \qquad \sum_{i=1}^{n} N_i = N$$

is isomorphic to the Higgs branch of



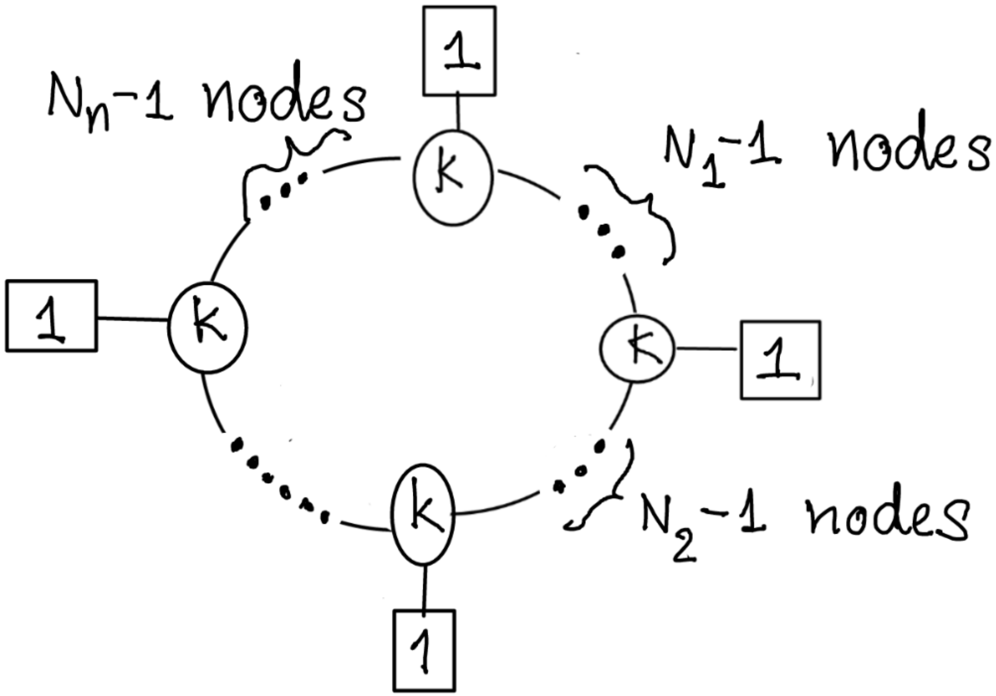
(n circular nodes)

- Problem: Find a quiver whose Coulomb branch describes such a moduli space of instantons.

The moduli space of k $SU(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$ such that

$SU(N) \to (U(N_1)\times U(N_2)\times \ldots \times U(N_n))/U(1)$, $\quad \sum_{i=1}^{n} N_i = N$

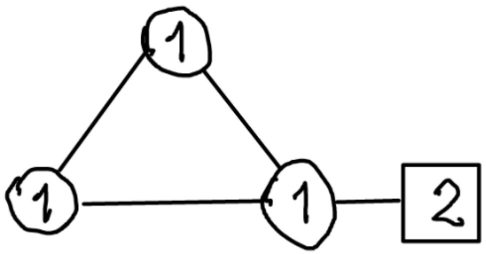
$\equiv$ the Coulomb branch of



(N circular nodes in total)

[Hanany, Witten '96; de Boer, Hori, Ooguri, Oz '96; Porrati, Zaffaroni '96]

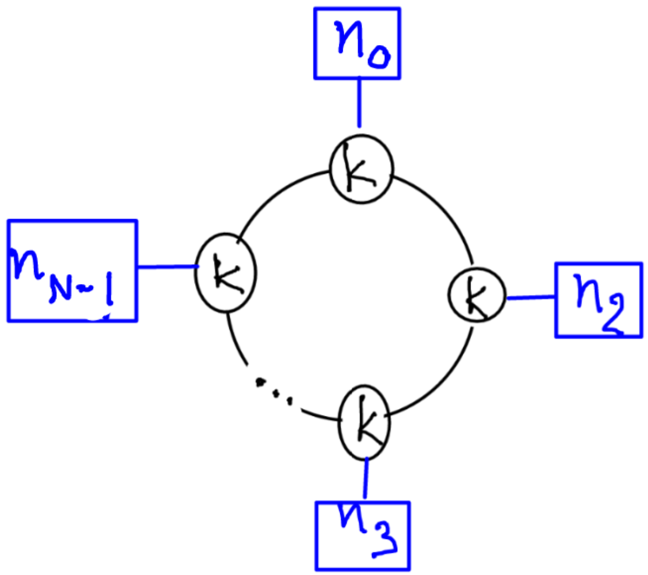
- <u>Examples</u>: Two theories that are mirror duals.

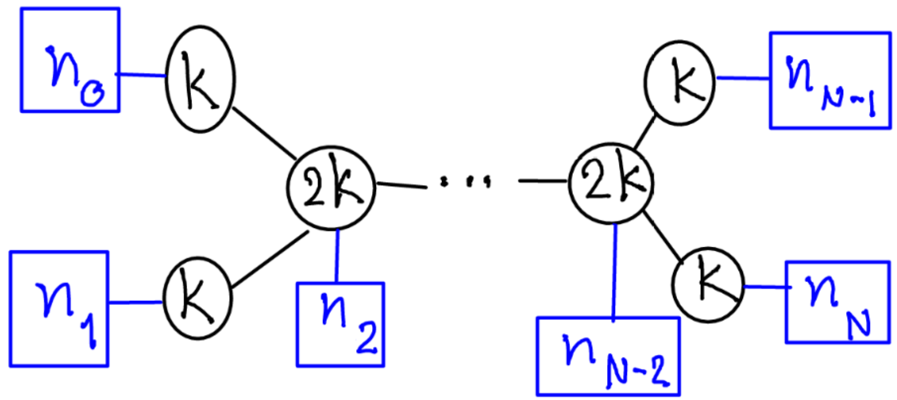
Quiver	Higgs	Coulomb
	1 $SU(2)$ inst. on $\mathbb{C}^2/\mathbb{Z}_3$ s.t. $SU(2)$ is unbroken $\left[\cong \mathbb{C}^2/\mathbb{Z}_3 \times \mathcal{N}^{\text{min}}_{SU(2)} \cong \mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}^2/\mathbb{Z}_2\right]$	1 $SU(3)$ inst. on $\mathbb{C}^2/\mathbb{Z}_2$ s.t. $SU(3)$ is unbroken $\left[\cong \mathbb{C}^2/\mathbb{Z}_2 \times \mathcal{N}^{\text{min}}_{SU(3)}\right]$
	1 $SU(3)$ inst. on $\mathbb{C}^2/\mathbb{Z}_2$ s.t. $SU(3)$ is unbroken $\left[\cong \mathbb{C}^2/\mathbb{Z}_2 \times \mathcal{N}^{\text{min}}_{SU(3)}\right]$	1 $SU(2)$ inst. on $\mathbb{C}^2/\mathbb{Z}_3$ s.t. $SU(2)$ is unbroken $\left[\cong \mathbb{C}^2/\mathbb{Z}_3 \times \mathcal{N}^{\text{min}}_{SU(2)}\right]$

- Can be realised on an M2-brane probing two ALE singularities. [Here: $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2/\mathbb{Z}_3$]
- Mirror symmetry exchanges the gauge group and the group corresponding to the orbifold.
[Porrati, Zaffaroni '96]

G-instantons on $\mathbb{C}^2/\mathbb{Z}_n$ (any simple group G)

- Can be realised from the Coulomb branch of the affine Dynkin diagram of G with various flavour nodes attached.





- What's the correspondence between the instanton configuration & the ranks n_i ?

- Rule 1 : The flavour node attached to the i^{th} gauge node is $U(n_i)$ such that

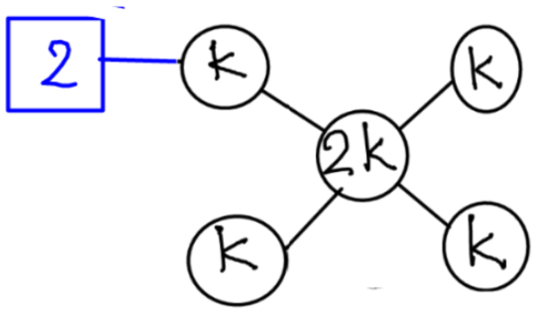
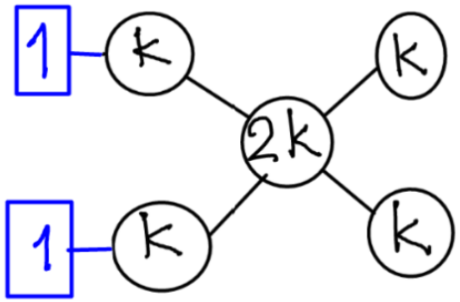
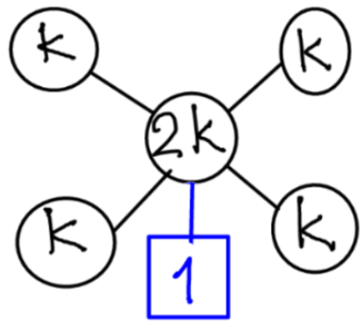
$$\text{order of orbifold} = n = \sum_{i=1}^{rk G} a_i n_i \qquad \left[\begin{array}{l} a_i = \text{coxeter label} \\ \text{of the } i^{th} \text{ gauge node} \end{array}\right]$$

- Rule 2 : The holonomy at infinity breaks G to a subgroup $H \times U(1)^{rkG - rkH}$, where H is a product of groups obtained by removing all nodes in the affine Dynkin diagram that have flavour nodes $n_i \neq 0$ attached

- If G is a simply laced group ($SU(N)$, $SO(2N)$, $E_{6,7,8}$), the Higgs branch of such a quiver diagram is the moduli space of $SU(n)$ instantons on $\mathbb{C}^2/\Gamma_G$. The breaking pattern of $SU(n)$ can be read off directly from the diagram.

Example 1: k $SO(8)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$

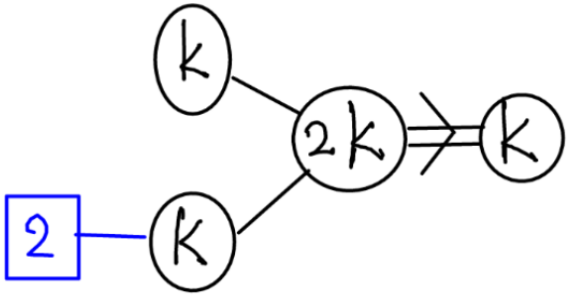
k $SU(2)$ instantons on $\mathbb{C}^2/\hat{D}_4$

			
Coulomb	k $SO(8)$ inst. $\mathbb{C}^2/\mathbb{Z}_2$ $SO(8)$ unbroken $\left[\overset{k=1}{\cong} \mathbb{C}^2/\mathbb{Z}_2 \times \mathcal{N}^{\min}_{SO(8)}\right]$	k $SO(8)$ inst. $\mathbb{C}^2/\mathbb{Z}_2$ $SO(8) \to SO(6) \times SO(2)$	k $SO(8)$ inst. $\mathbb{C}^2/\mathbb{Z}_2$ $SO(8) \to SO(4) \times SO(4)$
Higgs	k $SU(2)$ inst. $\mathbb{C}^2/\hat{D}_4$ $SU(2)$ unbroken $\left[\overset{k=1}{\cong} \mathbb{C}^2/\hat{D}_4 \times \mathcal{N}^{\min}_{SU(2)}\right]$	k $SU(2)$ inst. $\mathbb{C}^2/\hat{D}_4$ $SU(2) \to U(1)$	k $SU(2)$ inst. $\mathbb{C}^2/\hat{D}_4$ $SU(2) \to \phi$ $\left[\overset{k=1}{\cong} \mathbb{C}^4/\Gamma_{32}\right]^*$

* Γ_{32} is a finite gp. of order 32 [Thanks to M. Rocek & A. Hanany].

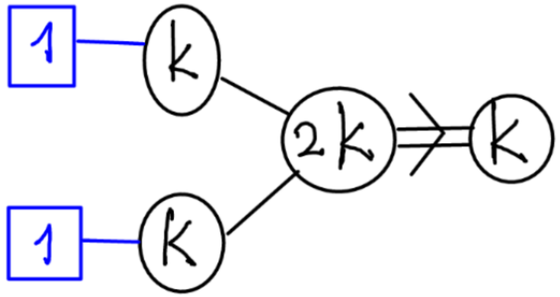
Example 2: k $SO(7)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$

Coulomb branch of:

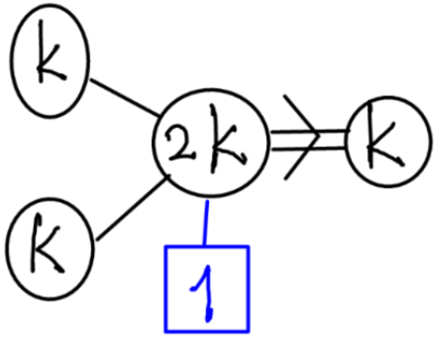


$SO(7)$ unbroken

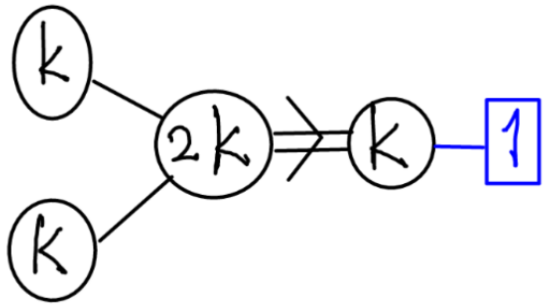
[Coulomb br. $\cong \mathbb{C}^2/\mathbb{Z}_2 \times \mathcal{N}^{\min}_{SO(7)}$
for $k=1$]



$SO(7) \to SO(5) \times SO(2)$



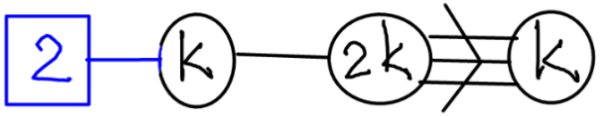
$SO(7) \to SO(3) \times SO(4)$



$SO(7) \to O(1) \times SO(6)$

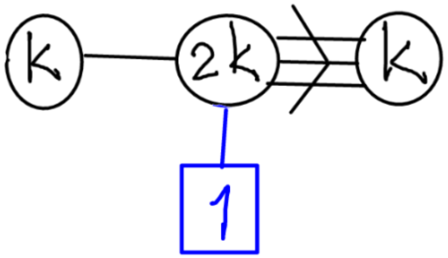
Example 3 : k G_2 instantons on $\mathbb{C}^2/\mathbb{Z}_2$

Coulomb branch of :



G_2 unbroken

$$\left[\begin{array}{l} \text{Coulomb br.} \\ \text{for } k=1 \end{array} \cong \mathbb{C}^2/\mathbb{Z}_2 \times \mathcal{N}^{\min}_{G_2} \right]$$



$G_2 \to SU(2) \times SU(2)$

Conclusions:

We presented 3d $\mathcal{N}=4$ quiver diagrams whose

Coulomb branch $\cong$ the moduli space of G-instantons on $\mathbb{C}^2/\mathbb{Z}_n$ (any simple group G)

Higgs branch $\cong$ the moduli space of $SU(N)$ inst. on $\mathbb{C}^2/\mathbb{Z}_n$, $\mathbb{C}^2/\hat{D}_{n+2}$, $\mathbb{C}^2/\hat{E}_{6,7,8}$

Future direction:

The moduli space of $SO(N)$, $E_{6,7,8}$ instantons on $\mathbb{C}^2/\hat{D}_{n+2}$, $\mathbb{C}^2/\hat{E}_{6,7,8}$.

PART III

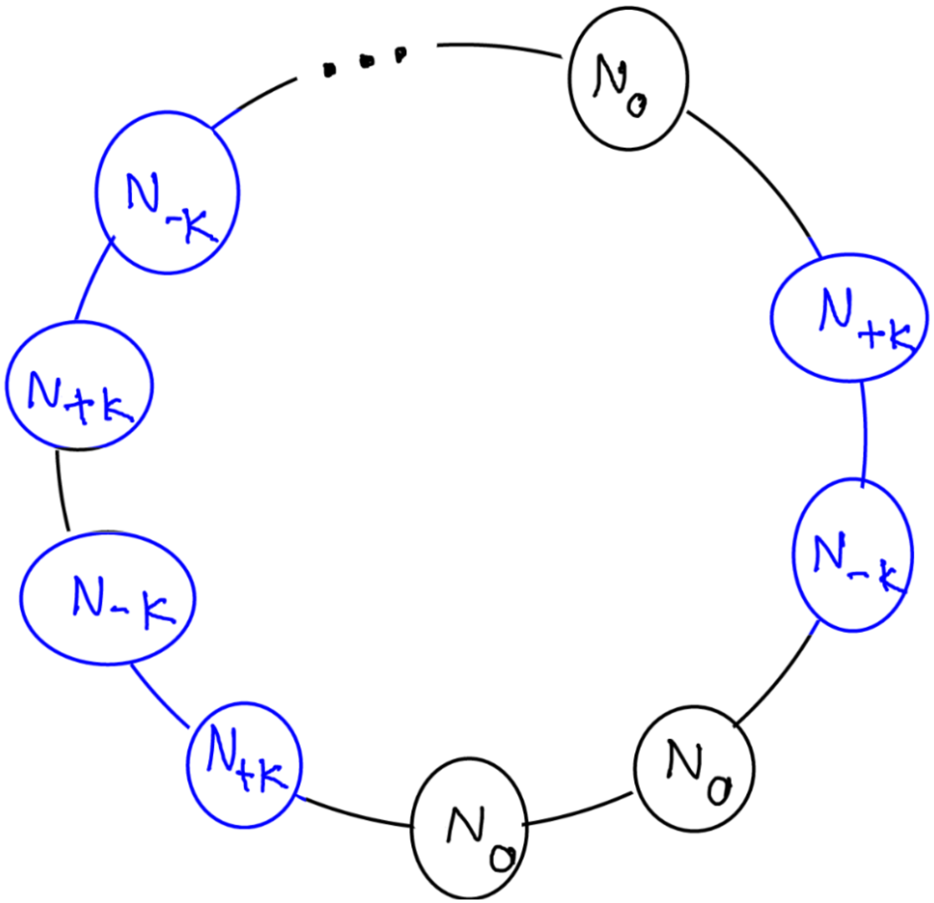
3d $\mathcal{N}=3$ gauge theories and the moduli space of $SU(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$

3d $\mathcal{N}=3$ gauge theories

- Obtained by adding Chern-Simons (CS) coupling to $\mathcal{N}=4$ theories
- The moduli space is parametrised by the monopole operators dressed by matter fields satisfying the BPS eqns.
- Consider theories on N M2 branes probing an orbifold of $\mathbb{C}^2/\mathbb{Z}_m \times \mathbb{C}^2/\mathbb{Z}_n$

  [Jafferis, Tomasiello '08; Imamura, Kimura '08; Cremonesi, N.M., Zaffaroni WIP]
- One branch $\cong$ Sym^N(orbifold of $\mathbb{C}^2/\mathbb{Z}_m \times \mathbb{C}^2/\mathbb{Z}_n$)

  Another branch $\cong$ the moduli space of $SU(M)$ instantons on $\mathbb{C}^2/\mathbb{Z}_r$



p pairs of N_{+k} and N_{-k}

q nodes of N_0

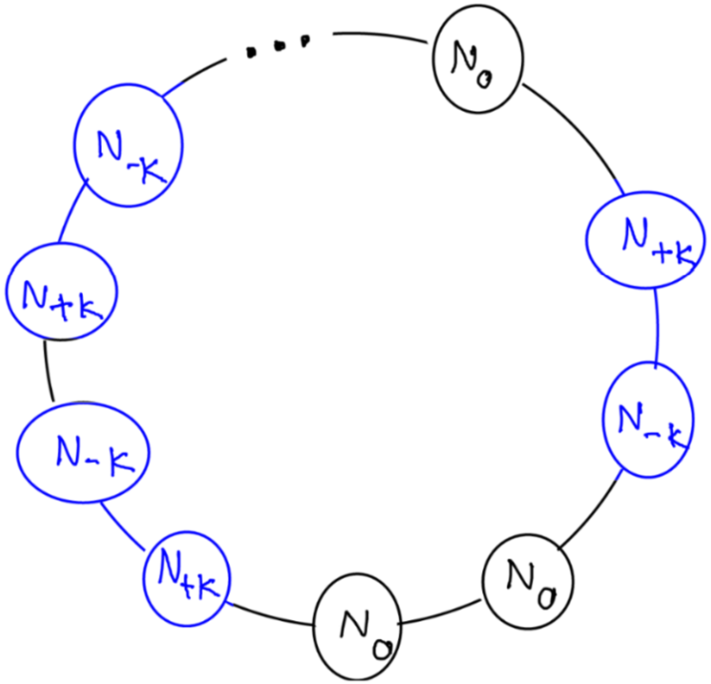
Total # of nodes $= 2p+q$.

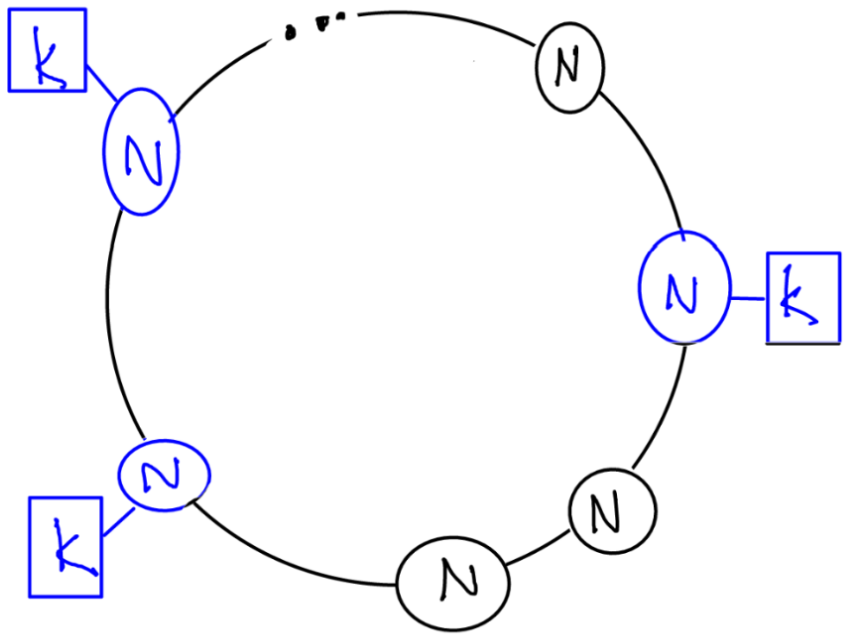
<u>Branch I</u>: The VEVs of all bi-fund hypers $\neq 0$.

$$\mathrm{Sym}^N\left(\left(\mathbb{C}^2/\mathbb{Z}_p \times \mathbb{C}^2/\mathbb{Z}_{(p+q)}\right)/\mathbb{Z}_k\right)$$

<u>Branch II</u>: All bifunds N_{+k} — N_{-k} are non-zero; other bifunds are zero.

Branch II of $\mathcal{N}=3$ $=$ Coulomb branch of $\mathcal{N}=4$ quiver



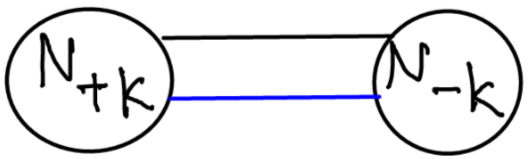


- Replace (N_0) by (N); (N_{+k})—(N_{-k}) by (N)—$\boxed{k}$

- $\boxed{\text{The moduli space of } N \ SU(p+q) \text{ inst. on } \mathbb{C}^2/\mathbb{Z}_{kp}}$

- For $k=1$, these two theories are related by an $SL(2,\mathbb{Z})$ transformation. [Gaiotto, Witten '08; Assel '14]

Example: The ABJM theory



Branch I: VEVs of both hypers $\neq 0$

$$\mathrm{Sym}^N(\mathbb{C}^4/\mathbb{Z}_k)$$

Branch II: VEV of ——— $\neq 0$
VEV of ——— $= 0$ (half-ABJM)

The moduli space of N "points" on $\mathbb{C}^2/\mathbb{Z}_k$

$$\cong \mathrm{Sym}^N(\mathbb{C}^2/\mathbb{Z}_k)$$

Open problem:

The moduli space of instantons in other gauge groups on general types of orbifolds from 3d $\mathcal{N}=3$ CS theories.

THANK YOU!