3d SUPERSYMMETRIC GAUGE THEORIES AND HILBERT SERIES

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Based on:

arXiv:	1309.2657	with	Hanany, Zaffaroni	3d $\mathcal{N}=4$
	1403.0585 2384		H, Mekareeya, Z	
	1408.6835		Ferlito, H, M	
	1410.1584		H, M, Z	
	1505.02409			3d $\mathcal{N}=2$ (subclass)
	1607.?		M, Z	3d $\mathcal{N}=3,2$

MODULI SPACE OF SUPERSYMMETRIC VACUA $\mathcal{M}$

[Luty, Taylor '95]

- **4d N=1 GAUGE THEORY:**
  - Gauge group G — VECTOR MULTIPLETS V^a (real)
  - Representation R — CHIRAL MULTIPLETS X^i (cplx)
  - G-invariant poly $W(X)$ — SUPERPOTENTIAL

- Scalar potential: $V = \sum_i |F_i|^2 + \frac{1}{2} g^2 \sum_a D^a D^a$

  $F_i = \frac{\partial W}{\partial X^i}$ "F-TERMS"

  $D^a = \sum_i X_i^\dagger (T^a)^i{}_j X^j$ "D-TERMS"

  $\left(D = \sum_i q_i |X_i|^2 - \xi \quad \text{for } G = U(1) \right)$

- **MODULI SPACE OF SUSY VACUA:**

$$\boxed{\mathcal{M} = \left\{ X, X^\dagger \;\middle|\; F_i(x) = 0 \;\forall i,\; D^a(X, X^\dagger) = 0 \;\forall a \right\} \Big/ G \;\cong\; \left\{ X \;\middle|\; \underset{\overset{\parallel}{\mathcal{F}}}{\partial W(X) = 0} \right\} \Big/ G^{\mathbb{C}}}$$

symplectic quotient — holomorphic quotient

4d N=1 SUSY $\Longrightarrow$ $\mathcal{M}$ KÄHLER (Today: $\mathcal{M}$ as cplx algebraic variety)

THE CHIRAL RING

[Lerche, Vafa, Warner '89]

• GAUGE INVARIANT CHIRAL OPERATORS $\mathcal{O}_i(x)$: $\boxed{\bar{Q}_{\dot\alpha}\mathcal{O}_i(x)=0}$ (Q_α, $\bar{Q}_{\dot\alpha}$ supercharges)

Expectation values $\langle \mathcal{O}_i \rangle$ are HOLOMORPHIC FUNCTIONS on $\mathcal{M}$.

$\langle \bar{Q}_{\dot\alpha}(\dots)\rangle = 0 \Rightarrow$

1) $\langle \mathcal{O}_i \rangle$ only depend on cohomology class
2) $\langle \mathcal{O}_{i_1}(x_1)\dots\mathcal{O}_{i_n}(x_n)\rangle = \langle \mathcal{O}_{i_1}\rangle\dots\langle\mathcal{O}_{i_n}\rangle$
3) Gauge invariant monomials involving F-terms are trivial

• CHIRAL RING: $\boxed{\mathcal{O}_i\mathcal{O}_j = c_{ij}{}^k\,\mathcal{O}_k + \underbrace{\bar{Q}_{\dot\alpha}(\dots)^{\dot\alpha}}}$ Drop from now on!

→ COORDINATE RING of $\mathcal{M}$: $\mathbb{C}[\mathcal{O}_1,\dots,\mathcal{O}_n]/I$

– GENERATORS
– RELATIONS

4d N=1 gauge theories: $\mathcal{O}_i$ are G– invariant polynomials in X.

THE HILBERT SERIES OF $\mathcal{M}$

[Pouliot '99; Benvenuti, Feng, Hanany, He '06]

$$\boxed{H(t,\hat{x}) = \mathrm{Tr}_{\mathcal{H}}\left(t^{R}\prod_{\hat{i}}\hat{x}_{\hat{i}}^{\hat{Q}_{\hat{i}}}\right)}$$

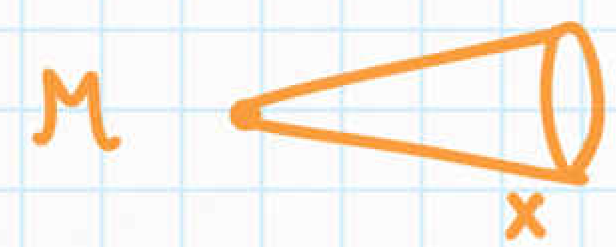
R : R-charge $\qquad [R,\bar{Q}]=\bar{Q}, \quad RW=2W.$

$\hat{Q}_{\hat{i}}$: FLAVOR charges $\qquad [\hat{Q}_{\hat{i}},\bar{Q}]=0, \quad \hat{Q}_{\hat{i}}W=0.$

$$\mathcal{H}=\bigoplus_r \mathcal{H}_r, \qquad \mathcal{H}_r=\{\mathcal{O}_i \mid \bar{Q}\mathcal{O}_i=0,\ R\mathcal{O}_i=r\,\mathcal{O}_i\}$$

* EXAMPLES $(G=\{1\} \Rightarrow \mathcal{M}=\mathcal{F})$

- 1 chiral X w/ $W=0$:



$\mathbb{C}[X]$

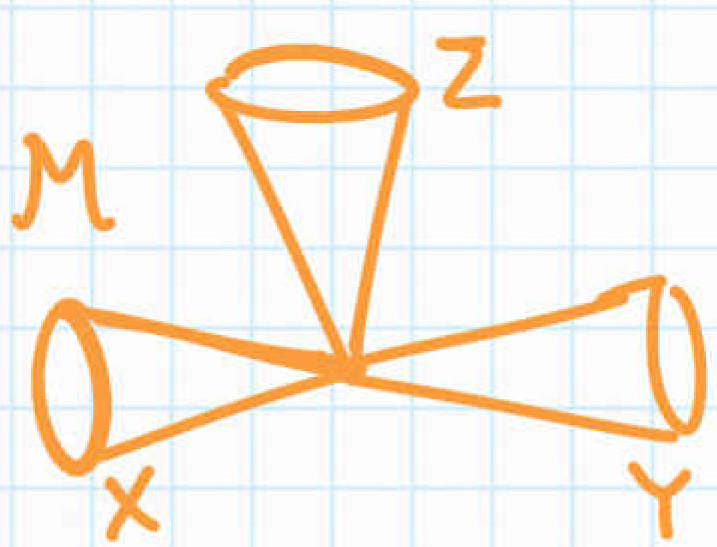
$$H = 1+\tau+\tau^2+\ldots = \frac{1}{1-\tau} = \mathrm{PE}[\tau] \qquad \tau = t^{R[X]}$$

- 1 chiral X w/ $W=X^{N+1}$:

$\mathcal{M}$ • $\qquad \mathbb{C}[X]/\langle X^N\rangle$

$$H = 1+\tau+\tau^2+\ldots+\tau^{N-1} = \frac{1-\tau^N}{1-\tau} = \mathrm{PE}[\tau-\tau^N] \qquad \tau=t^{\frac{2}{N+1}}$$

- 3 chirals X,Y,Z w/ $W=XYZ$:



$$\frac{\mathbb{C}[X,Y,Z]}{\langle YZ, ZX, XY\rangle}$$

$$H = \frac{1}{1-\tau x}+\frac{1}{1-\tau y}+\frac{1}{1-\tau z}-2 = \qquad \tau=t^{2/3}, \quad xyz=1$$

$$= \frac{1-\tau^2(yz+xz+xy)+2\tau^3 xyz}{(1-\tau x)(1-\tau y)(1-\tau z)}$$

• GAUGING G :

$$\boxed{H_{\mathcal{M}}(t,\hat{x}) = \oint d\mu_G(x)\; H_{\mathcal{F}}(t,\hat{x},x)} \quad , \qquad \oint d\mu_G(x) \equiv \left(\prod_{j=1}^{r} \oint_{|x_j|=1} \frac{dx_j}{2\pi i x_j} \right) \prod_{\alpha\in\Delta_+} \left(1 - \prod_{i=1}^{r} x_i^{\alpha_i}\right)$$

$r \equiv \mathrm{rk}(G)$

* EXAMPLES

• G with 1 adjoint Φ :

(→ Coulomb branch of 4d $\mathcal{N}=2$)

$$H(\tau) = \oint d\mu_G(x)\; \mathrm{PE}\left[\tau \chi^G_{ad}(x)\right] = \prod_{i=1}^{r} \frac{1}{1-\tau^{d_i(G)}}$$

Casimir invariants

$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[u_1,\dots,u_r]$

$\deg u_i = d_i(G)$

• $G=U(1)$ with 2 flavors:

$$\begin{array}{ccccc} \boxed{2} & \rightarrow & \textcircled{1} & \rightarrow & \boxed{2} \\ v & & x & & u \end{array}$$

$$H_B(\tau,u,v) \equiv g_1(\tau,u,v;B) = \oint \frac{dx}{2\pi i x}\, x^{-B}\, \mathrm{PE}\left[\tau x\left(u+\tfrac{1}{u}\right) + \tau \tfrac{1}{x}\left(v+\tfrac{1}{v}\right)\right]$$

ξ

$$= \begin{cases} \displaystyle\sum_{n=0}^{\infty} [n+B;n]_{u,v}\, \tau^{2n+B} , & B \geq 0 \\ \displaystyle\sum_{n=0}^{\infty} [n;n+|B|]_{u,v}\, \tau^{2n+|B|} , & B \leq 0 \end{cases}$$

$B=0$: $\quad H(\tau,u,v;0) = \mathrm{PE}\left[\,[1;1]_{u,v}\tau^2 - \tau^4\right]$ $\qquad$ Conifold $\quad \dfrac{\mathbb{C}[X,Y,Z,W]}{\langle XY-ZW\rangle}$

$$\mathrm{PE}[f(t_1,\dots,t_n)] \equiv \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f(t_1^k,\dots,t_n^k)\right) \Rightarrow \mathrm{PE}\left[\sum_i a_i \prod_j t_j^{b_{ij}}\right] = \prod_i \left(1-\prod_j t_j^{b_{ij}}\right)^{-a_i}$$

$$[\alpha;\beta]_{u,v} \equiv \chi_{[\alpha]}(u)\,\chi_{[\beta]}(v)$$

- Moduli spaces of vacua of 4d ($\mathcal{N}\geq 1$) SUSY gauge theories and Hilbert series:

  MATH — (Hyper) Kähler quotients

  PHYS — mostly CLASSICAL (quantum: [Seiberg '93-'94; ...])

- In the rest of my talk, I will discuss a different construction

  MATH — (Hyper) Kähler

  PHYS — QUANTUM

which is based on the physics of 3d ($\mathcal{N}\geq 2$) SUSY gauge theories.

[Intriligator, Seiberg '96; de Boer, Hori, Ooguri, Oz '96; Aharony, Hanany, Intriligator, Seiberg, Strassler '97; de Boer, Hori, Oz '97; Dorey, Tong '99; Tong '00; Intriligator, Seiberg '13 ...]

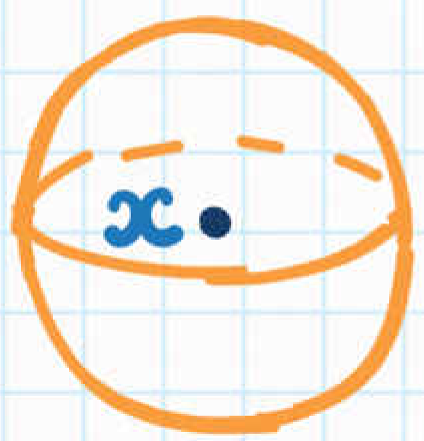
NOVELTY — 't Hooft monopole operators: extra chiral operators subject to quantum relations not derived from a superpotential.

'T HOOFT MONOPOLE OPERATORS

['t Hooft '78; Borokhov, Kapustin, Wu '02]

• 3d $\mathcal{N}=2$ "BARE" MONOPOLE OPERATOR $V_m(x)$:

Euclidean path integral over field config's with $\frac{1}{2}$-BPS Dirac monopole singularity at x: $U(1) \hookrightarrow G$.



(r, θ, φ)

$$\boxed{\begin{aligned} A_\pm &\underset{r\to 0}{\sim} \frac{m}{2}(\pm 1 - \cos\theta)\, d\varphi \\ \sigma &\sim \frac{m}{2r} \end{aligned}}$$

VECTOR MULTIPLET $\rightsquigarrow$ CHIRAL MULTIPLETS

$\underbrace{A_\mu, \sigma, \ldots}_{\mathbb{R}} \qquad V_m$

$m \in \underset{\text{Cartan}}{\mathfrak{h}} / \underset{\text{Weyl}}{W}$ MAGNETIC CHARGE

Dirac quantization $\boxed{e^{2\pi i m} = \mathbb{1}_G} \Rightarrow \boxed{m \in \Gamma_{G^\vee}/W}$.

[Goddard, Nuyts, Olive '77]

• Can do the same for GLOBAL non-R SYMMETRIES : BACKGROUND MAGNETIC CHARGES.

Monopole operators $V_{m;\hat{m},B}$

m	$\longleftrightarrow \sigma$	real scalar	GAUGE
$\hat{m}$	$\longleftrightarrow \hat{\sigma}$	real mass	FLAVOR
B	$\longleftrightarrow \xi$	FI parameter	TOPOLOGICAL

CHARGES OF BARE MONOPOLE OPERATORS

CLASSICAL:

- TOPOLOGICAL SYMMETRY $G_J = Z(G^\vee)$

For $G = U(N)$: $\boxed{J(m) = \mathrm{Tr}\, m = \sum_{i=1}^{N} m_i}$ $\quad (m_i) \in \mathbb{Z}^N / S_N$

- CHARGES FROM CHERN-SIMONS COUPLINGS e.g. $\frac{k_{AB}}{4\pi} \int A_A \wedge dA_B + \dots$ for $U(1)$'s

$$\{M_A\} \equiv \{\underbrace{m_i}_{\text{gauge}}, \underbrace{\hat{m}_{\hat{i}}}_{\text{flavor}}, \underbrace{B_i}_{\text{topological}}, \underbrace{0}_{R}\}$$

$$\boxed{Q_A^{cl}(M) = -\sum_B k_{AB} M_B}$$

QUANTUM:

$$\boxed{Q_A^{q}(M) = -\frac{1}{2} \sum_{\text{fermi } \psi_a} Q_A[\psi_a] \, |m_a^{eff}(M)|}$$

[Borokhov, Kapustin, Wu '02
Imamura, Yokoyama '11
Benini, Closset, SC '11]

$$m_a^{eff}(M) \equiv \sum_A Q_A[\psi_a] M_A$$ "EFFECTIVE MASS"

We will count monopole operators according to their (classical + quantum) charges:

$$\boxed{Q_A(M) = -\sum_B k_{AB}^{eff}(M)\, M_B}\ , \quad \text{where} \quad \boxed{k_{AB}^{eff}(M) = k_{AB} + \frac{1}{2} \sum_{\text{fermi } \psi_a} Q_A[\psi_a] Q_B[\psi_a] \, \mathrm{sign}\, m_a^{eff}(M)}\ .$$

EFFECTIVE CS LEVELS

DRESSED MONOPOLE OPERATORS

[SC, Hanany, Zaffaroni '13
SC '15; SC, Mekareeya, " '16]

- BARE monopole operator defined via VECTOR multiplets: ← Gauge/Flavor/Top.

$G \times \hat{G} \times G_J$

$G \xrightarrow{m} G_m$ "RESIDUAL GAUGE GROUP"

$G_m \cdot m = 0$

$$\longrightarrow \prod_{\alpha \in \Delta_+} t^{-|\alpha(m)|} (1 - x^\alpha)^{\delta_{\alpha(m),0}}$$

- Can be DRESSED by matter fields (CHIRAL multiplets) s.t. ← Representation

$(R, \hat{R}, 1)$

$\boxed{m^{\text{eff}}_{\rho,\hat\rho}(m,\hat m) = \rho(m) + \hat\rho(\hat m) = 0}$. "RESIDUAL MATTER FIELDS"

$$\longrightarrow \prod_{\rho,\hat\rho} \left(t^{r-1} x^\rho \hat{x}^{\hat\rho}\right)^{-\frac{1}{2}|\rho(m)+\hat\rho(\hat m)|} \mathrm{PE}\left[\delta_{\rho(m)+\hat\rho(\hat m),0}\, t^r x^\rho \hat{x}^{\hat\rho}\right]$$ (if $W=0$)

Dress bare monopole op. by G_m-invariants of residual matter fields, then average over W.

* Kronecker δ: adjoint Higgs mechanism.

* Prefactor: quantum correction to monopole charges

DRESSED MONOPOLE OPERATORS AND HILBERT SERIES

DRESSING governed by "RESIDUAL GAUGE THEORY" $T_{m;\hat{m},B}$:

- RESIDUAL GAUGE GROUP G_m (FLAVOR GROUP $\hat{G}_{\hat{m}}$)
- RESIDUAL MATTER FIELDS in rep. of $G_m \times \hat{G}_{\hat{m}}$
- RESIDUAL SUPERPOTENTIAL
- BACKGROUND ELECTRIC CHARGES $Q_i(m,\hat{m},B)$ for G_m

$\rightarrow$ Hilbert series $H^{T_{m;\hat{m},B}}_{Q(m,\hat{m},B)}(t,\hat{x})$

"DRESSING FACTOR"

• HILBERT SERIES :

$$H(t,\hat{x},z;\hat{m},B) = \mathrm{Tr}_{\mathcal{H}_{\hat{m},B}}\left(z^J t^R \prod_{\hat{i}} \hat{x}_{\hat{i}}^{\hat{Q}_{\hat{i}}}\right)$$

[SC '15; SC, Mekareeya, Zaffaroni '16 to appear]

$$\boxed{H(t,\hat{x},z;\hat{m},B) = \sum_{m\in\Gamma_q} z^{J(m)} t^{R(m,\hat{m},B)} \prod_{\hat{i}} \hat{x}_{\hat{i}}^{\hat{Q}_{\hat{i}}(m,\hat{m},B)} \cdot H^{T_{m;\hat{m},B}}_{Q(m,\hat{m},B)}(t,\hat{x})}$$

$\Gamma_q = \Gamma_{G^\vee}/\mathcal{W}$ or a sublattice thereof (due to nonperturbative effects).

"QUANTUM MAGNETIC LATTICE"

COULOMB BRANCH OF 3d $\mathcal{N}=4$ GAUGE THEORIES

G • $(\mathcal{N}=4 \text{ VECTOR}) = (\mathcal{N}=2 \text{ VECTOR}) \oplus (\mathcal{N}=2 \text{ CHIRAL } \Phi \in \text{adj})$ — $\mathcal{M}_C$: HyperKähler, $\dim_{\mathbb{H}} \mathcal{M}_C = r$

R • $(\mathcal{N}=4 \text{ HYPER}) = (\mathcal{N}=2 \text{ CHIRAL} \in R) \oplus (\mathcal{N}=2 \text{ CHIRAL} \in \overline{R})$ — $\mathcal{M}_H$: HK quotient

$$\boxed{R(m,\hat{m}) = -\frac{1}{2}\sum_{\alpha}|\alpha(m)| + \frac{1}{2}\sum_{\rho,\hat{\rho}}|\rho(m)+\hat{\rho}(\hat{m})|}$$

[Borokhov, Kapustin, Wu '02
Gaiotto, Witten '08; ...]

HILBERT SERIES of $\mathcal{M}_C$:

(good/ugly theories)
cf [Gaiotto-Witten '08]

$$\boxed{H(\tau, z; \hat{m}) = \sum_{m \in \Gamma_{G^\vee}/W} z^{J(m)} \tau^{2R(m,\hat{m})} P_G(\tau^2; m)}$$

[SC, Hanany, Zaffaroni '13
" + Mekareeya '14]

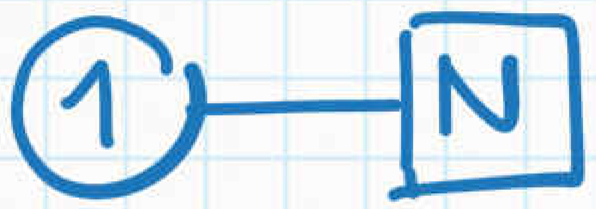
$(\tau^2 = t)$

$$P_G(t;m) = \prod_{i=1}^{r} \frac{1}{1-t^{d_i(m)}}$$ Casimirs of G_m

REMARKS:

- $\hat{m}=0$: $\mathcal{M}_C$ cone vs. $\hat{m}\neq 0$: (partial) resolution
- (Very computable) limit of SUSY indices [Razamat, Willett '14; Closset, Kim '16].
- Can often deduce charges of generators / relations. Complementary to [Bullimore, Dimofte, Gaiotto '15].

EXAMPLES $(\hat{m}=0)$

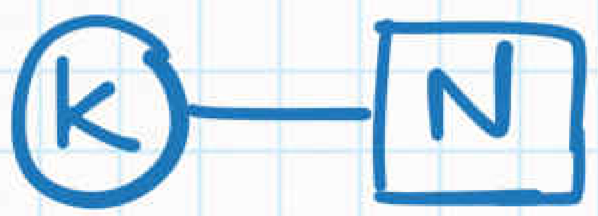


$$H(\tau,z)=\frac{1}{1-\tau^2}\sum_{m\in\mathbb{Z}} z^m \tau^{N|m|} = \frac{1-\tau^{2N}}{(1-\tau^2)(1-z\tau^N)(1-z^{-1}\tau^N)}$$

$\mathcal{M}_C = \mathbb{C}^2/\mathbb{Z}_N$

$\mathbb{C}[\Phi, V_+, V_-]/\langle V_+V_- - \Phi^N\rangle$

cf [Intriligator-Seiberg '96]



$$H(\tau,z)=\sum_{m_1\geq\ldots\geq m_k} z^{\sum_i m_i}\tau^{-2\sum_{i<j}|m_i-m_j|+N\sum_i|m_i|}P_{U(k)}(\tau^2;m)$$

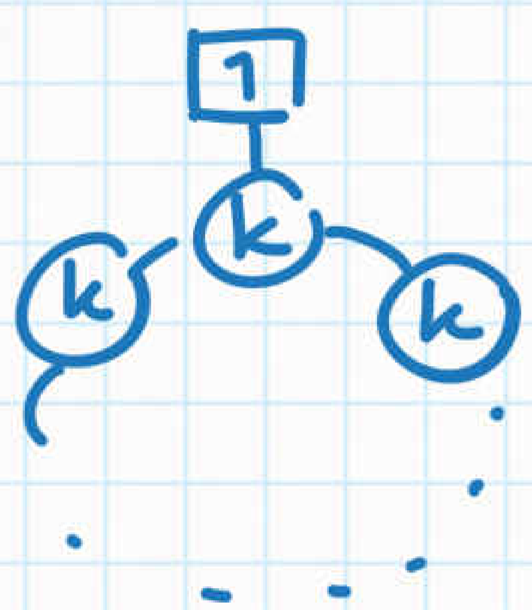
$$=\mathrm{PE}\left[\sum_{j=1}^{k}\left(\tau^{2j}+(z+z^{-1})\tau^{N-2(k+j)}-\tau^{2(N-k-j)}\right)\right]$$

3k generators
k relations

see also [Bullimore, Dimofte, Gaiotto '15]

• EXTENDED AFFINE "QUIVERS" for INSTANTON MODULI SPACES: [SC, Ferlito, Hanany, Mekareeya '14]

e.g.



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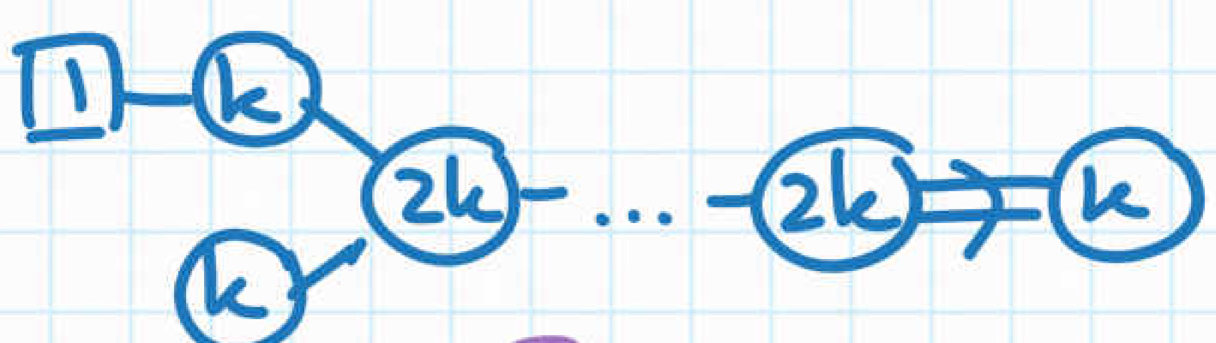

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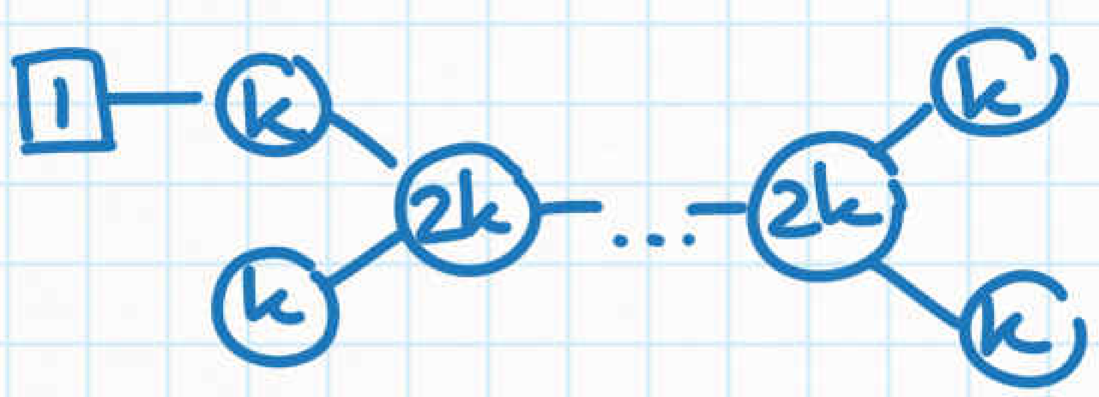


D

etc.

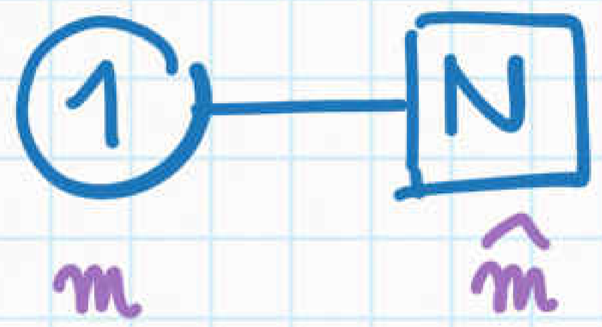
$$H_{\mathcal{M}_{G,k}}=\mathrm{PE}\Big[\sum_{p=0}^{k-1}\big(\underbrace{[p+1;}_{SU(2)}\,0]t^{p+1}+[p;\underbrace{Adj}_{G}]t^{p+2}\big)+$$

$$-\ldots t^{k+2}+\ldots\Big]$$

for any G

see [Braverman, Finkelberg, Nakajima '16]

BACKGROUND MAGNETIC CHARGES AND RESOLUTIONS



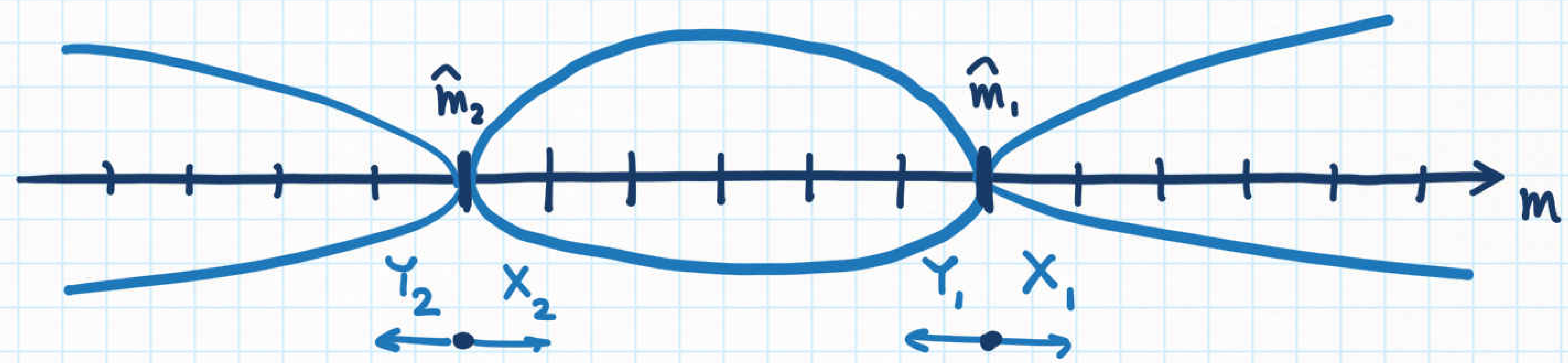
$$H(\tau, z; \hat{m}) = \sum_{m\in\mathbb{Z}} z^{m}\, \tau^{\sum_{a=1}^{N}|m-\hat{m}_a|}$$

Consider $N=2$, $\hat{m} = (\hat{m}_1, \hat{m}_2)$ with $\hat{m}_1 \geq \hat{m}_2$:

$$H(\tau, z; \hat{m}) = \mathrm{PE}[\tau^2]\cdot\left(z^{\hat{m}_2}\tau^{\hat{m}_1-\hat{m}_2}\,\mathrm{PE}[z^{-1}\tau^2] + z^{\hat{m}_2}\tau^{\hat{m}_1-\hat{m}_2}\cdot\sum_{\ell=0}^{\hat{m}_1-\hat{m}_2} z^{\ell} + z^{\hat{m}_1}\tau^{\hat{m}_1-\hat{m}_2}\,\mathrm{PE}[z\tau^2]\right)$$

$X_a Y_a = \Phi$

$X_{a+1} Y_a = 1$ $\quad\forall a$.



$V_{m=\hat{m}_2-q;\hat{m}} = V_{m=\hat{m}_2;\hat{m}}\, Y_2^{q}$

$V_{\hat{m}_2+\ell;\hat{m}} = V_{\hat{m}_2;\hat{m}}\, X_2^{\ell}$

$= V_{\hat{m}_1;\hat{m}}\, Y_1^{\hat{m}_1-\hat{m}_2-\ell}$

$V_{m=\hat{m}_1+p;\hat{m}} = V_{m=\hat{m}_1;\hat{m}}\, X_1^{p}$

$$\underset{\mathbb{C}^2}{(X_a, Y_a)} \longmapsto \begin{cases} V_+ = X_a^{a} Y_a^{a-1} \\ V_- = X_a^{N-a} Y_a^{N-a+1} \\ \Phi = X_a Y_a \end{cases}$$

$V_+ V_- = \Phi^N$

with transitions

$X_{a+1} Y_a = 1$.

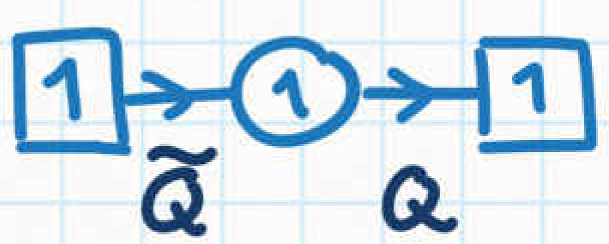
MODULI SPACES OF 3d $\mathcal{N}=2$ THEORIES

- **YANG-MILLS THEORIES** (W=0 for simplicity, $\hat{m}=B=0$) [SC '15; Hanany et al '15]

$$\boxed{H(t,z,\hat{x}) = \sum_{m\in\Gamma_q} z^{J(m)} \prod_{i=1}^{r} \oint \frac{dx_i}{2\pi i x_i} \prod_{\alpha\in\Delta_+} (1-x^\alpha)^{\delta_{\alpha(m),0}}\, t^{-|\alpha(m)|} \prod_{\rho,\hat{\rho}} \left(t^{r_{\rho\hat{\rho}}-1} x^\rho \hat{x}^{\hat{\rho}}\right)^{-\frac{1}{2}|\rho(m)|} \mathrm{PE}\left[\delta_{\rho(m),0}\, t^{r_{\rho\hat{\rho}}} x^\rho \hat{x}^{\hat{\rho}}\right]}$$

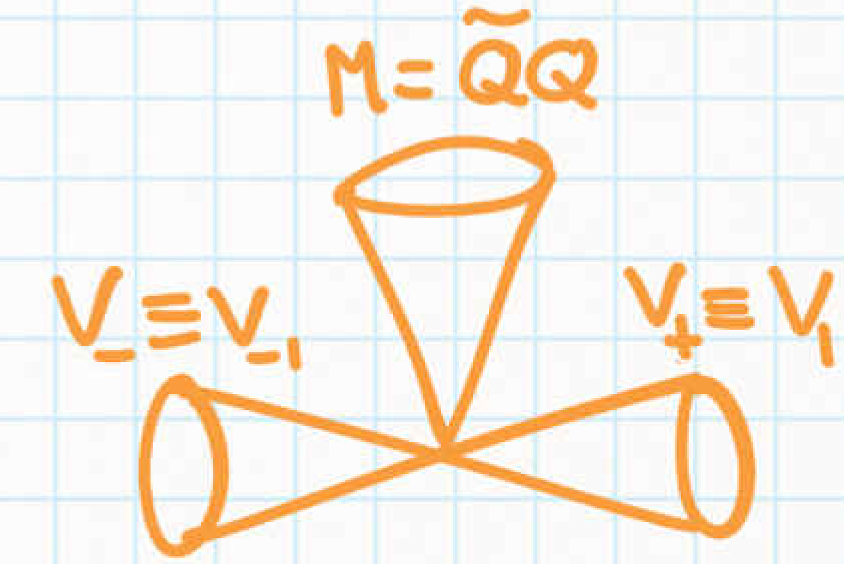
$$\Gamma_q = \left\{ m\in\Gamma_{G^\vee}/W \;\middle|\; \sum_\rho \rho(\alpha_i^\vee)\,\mathrm{sign}\,\rho(m) \neq 0 \;\vee\; \alpha_i(m)=0 \quad \forall\, i=1,\dots,r \right\}$$

* Examples



$$H(t;z,y) = \sum_{m\in\mathbb{Z}} \underset{\substack{\downarrow\\ U(1)_J}}{z^m}\; \underset{\substack{\downarrow\\ U(1)_R}}{t^{(1-r)|m|}}\; \underset{\substack{\downarrow\\ U(1)_A}}{y^{-|m|}} \oint \frac{dx}{2\pi i x} \mathrm{PE}\left[\delta_{m,0}\, t^r y \left(x+\frac{1}{x}\right)\right]$$

$$= \frac{1}{1-y^2t^{2r}} + \frac{1}{1-zy^{-1}t^{1-r}} + \frac{1}{1-z^{-1}y^{-1}t^{1-r}} - 2 .$$



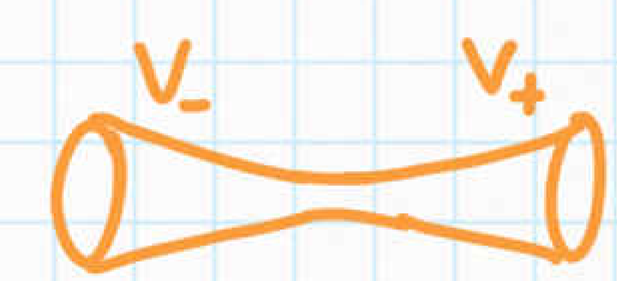
$\mathbb{C}[M,V_+,V_-]/\langle V_+V_-, V_\pm M\rangle$

cf [Aharony et al, de Boer et al '97]

Turning on $W = m_{\mathbb{C}}\tilde{Q}Q$:

$$H(t;z,y) = \sum_{m\in\mathbb{Z}} z^m \left(t^{1-r}y^{-1}\right)^{|m|} = \frac{1}{1-zy^{-1}t^{1-r}} + \frac{1}{1-z^{-1}y^{-1}t^{1-r}} - 1$$

$$= \mathrm{PE}\left[zy^{-1}t^{1-r} + z^{-1}y^{-1}t^{1-r} - y^{-2}t^{2(1-r)}\right]$$



$\mathbb{C}[V_+,V_-]/\langle V_+V_- - m_{\mathbb{C}}\rangle$

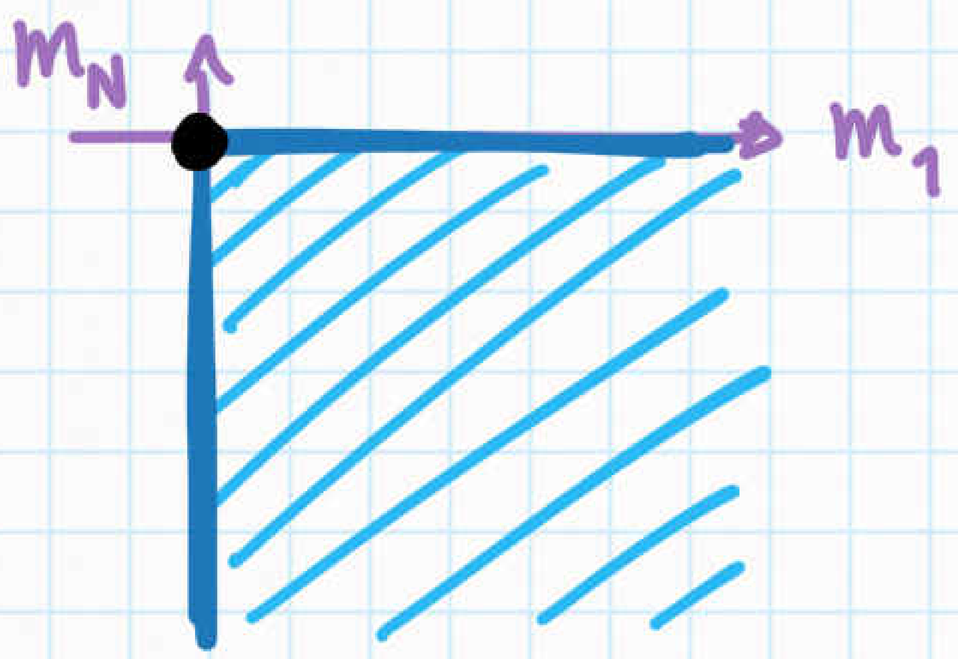
cplx structure deformation

* SQCD

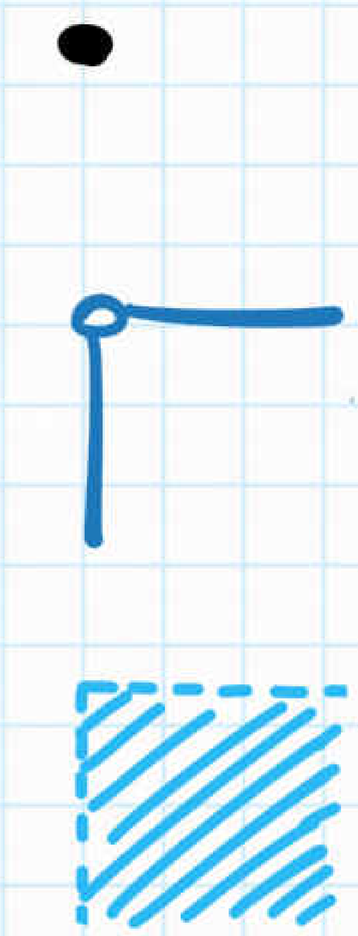
$$\boxed{N_f} \xrightarrow{\tilde{Q}} \textcircled{N} \xrightarrow{Q} \boxed{N_f}$$

$$\Gamma_q = \left\{ m = (m_1, 0, \dots, 0, m_N) \in \mathbb{Z}^N \,\middle|\, m_1 \geq 0 \geq m_N \right\}$$

due to instantons



- $m_1 = 0 = m_N$: $\quad T_m = [U(N) \text{ w/ } N_f]$
- $m_1 > 0 = m_N$: $\quad T_m = [U(N-1) \text{ w/ } N_f] \times U(1)$
  $m_1 = 0 > m_N$
- $m_1 > 0 > m_N$: $\quad T_m = [U(N-2) \text{ w/ } N_f] \times U(1)^2$



Just from this structure, it follows that the chiral ring is

$$\mathbb{C}[M_{N_f \times N_f}, V_+, V_-] \Big/ \left\langle \begin{array}{l} \text{minor}_{N+1}(M) = 0 \\ V_\pm \cdot \text{minor}_N(M) = 0 \\ V_+ V_- \cdot \text{minor}_{N-1}(M) = 0 \end{array} \right\rangle$$

cf [Aharony '97]

without assuming a singular dynamically generated

$$W_{dyn} = (V_+ V_- \det M)^{\frac{1}{N_f - N + 1}}.$$

• A FUN CHERN-SIMONS EXAMPLE: $SU(2)_{\frac{1}{2}}$ w/ 3 doublets

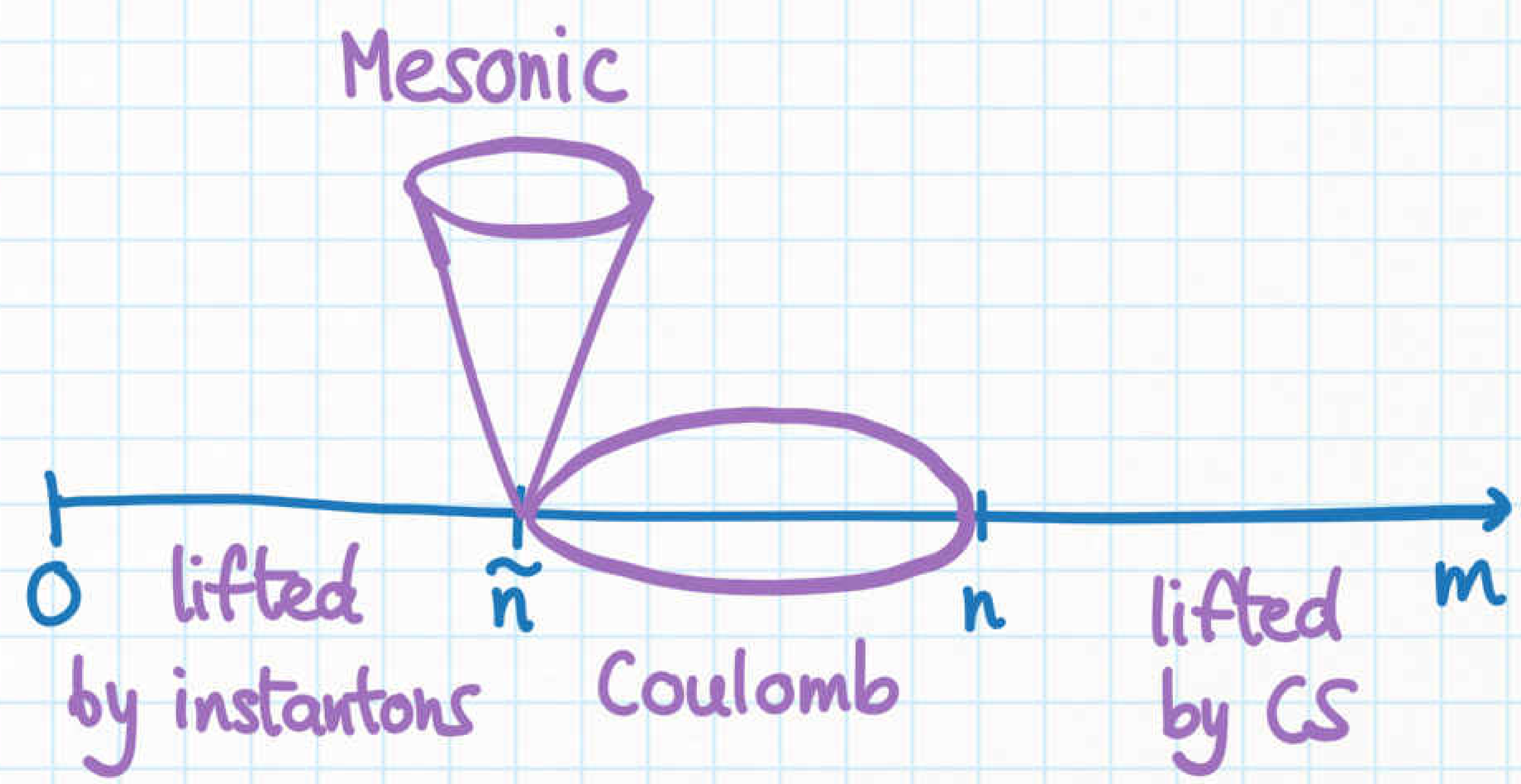
[Tong '00]

$$H(t,\hat{x}_{1,2,3};\hat{m}_{1,2,3}) = \sum_{m\in\Gamma_q} \oint \frac{dx}{2\pi i x}\, x^{-m}(1-x^2)^{\delta_{2m,0}}\, t^{-2m} \cdot \prod_{a=1}^{3}\prod_{s_a=\pm1}\left(t^{r-1}x^{s_a}\hat{x}_a\right)^{-\frac{1}{2}|s_a m+\hat{m}_a|}\, \mathrm{PE}\left[\delta_{s_a m+\hat{m}_a,0}\, t^r x^{s_a}\hat{x}_a\right]$$

• $\hat{m}_1=-n,\ \hat{m}_2=\tilde{n},\ \hat{m}_3=-\tilde{n}$ with $\underline{0\le\tilde{n}\le n}$:

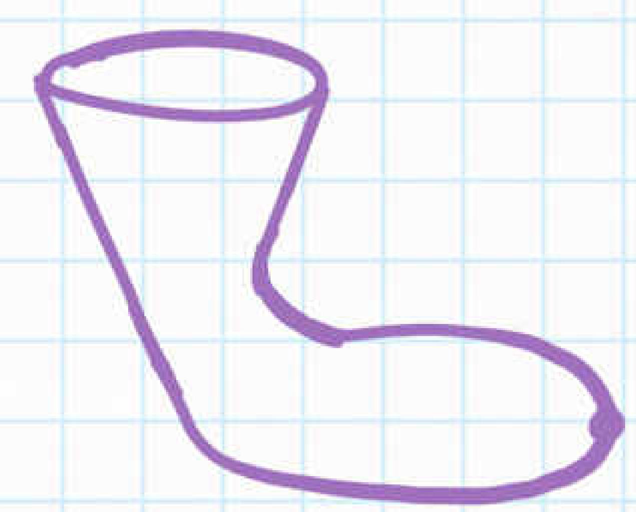
$$\Gamma_q=\mathbb{Z}_{\ge\tilde{n}}\ ,\qquad Q[V_{m;\hat{m}}]=\begin{cases}0 & ,\ 0\le m\le n\\ -m+n & ,\ n\le m\end{cases}$$

$\Rightarrow$ NAIVELY:



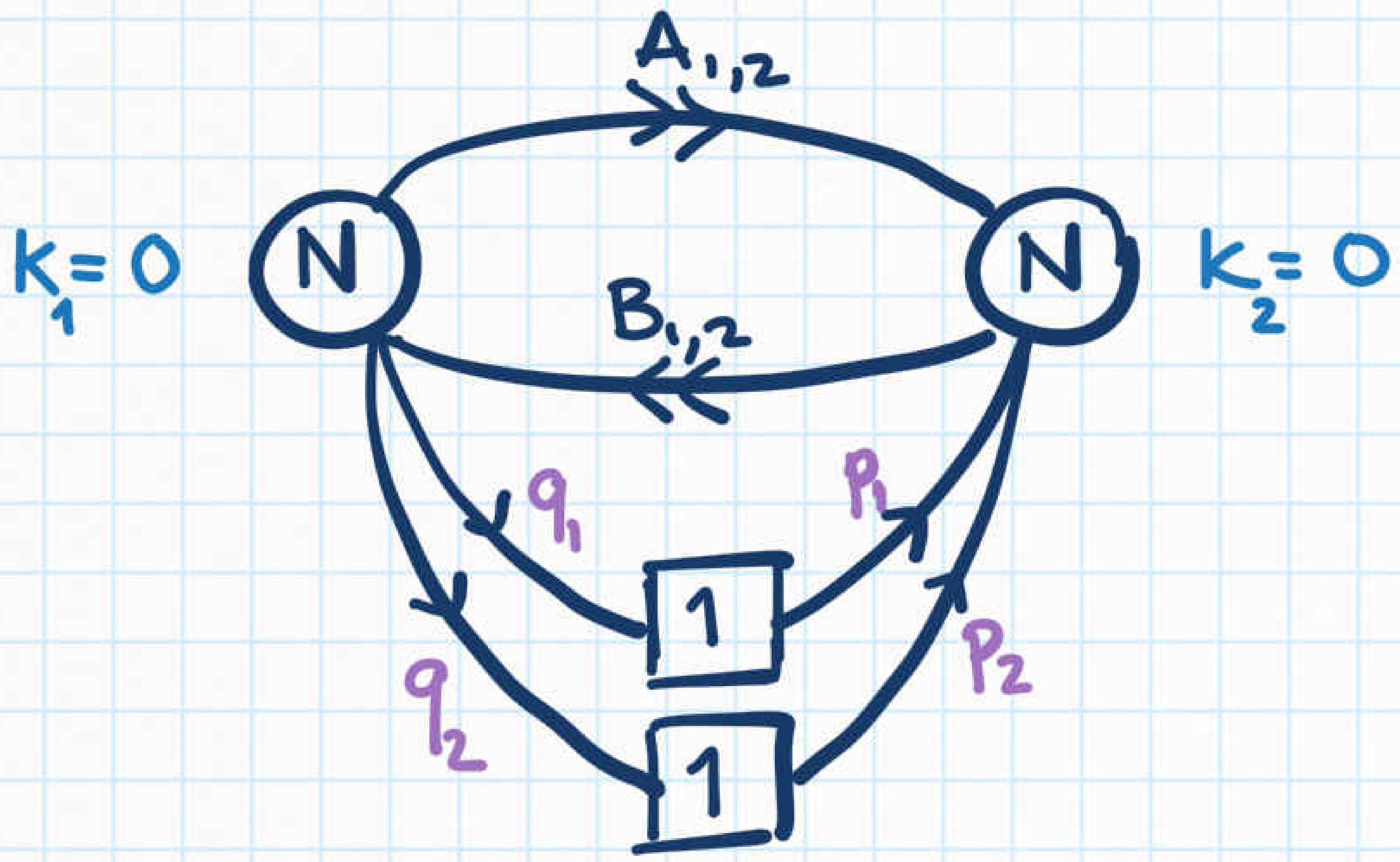
$$\begin{aligned}H(t,\hat{x}_{1,2,3};-n,\tilde{n},-\tilde{n}) &= t^{-2\tilde{n}}(t^{r-1}\hat{x}_1)^{-n}(t^{2(r-1)}\hat{x}_2\hat{x}_3)^{-\tilde{n}}\oint\frac{dx}{2\pi i x}\,\mathrm{PE}\left[t^r(x^{-1}\hat{x}_2+x\hat{x}_3)\right]+\\ &+\sum_{m=\tilde{n}}^{n} t^{-2m}(t^{r-1}\hat{x}_1)^{-n}(t^{2(r-1)}\hat{x}_2\hat{x}_3)^{-m}\\ &- t^{-2m}(t^{r-1}\hat{x}_1)^{-n}(t^{2(r-1)}\hat{x}_2\hat{x}_3)^{-\tilde{n}}\\ &= \left(t^{3(r-1)}\hat{x}_1\hat{x}_2\hat{x}_3\right)^{-n}\mathrm{PE}\left[t^{2r}\hat{x}_2\hat{x}_3\right]\end{aligned}$$

Mesonic
+
Coulomb
−
intersection

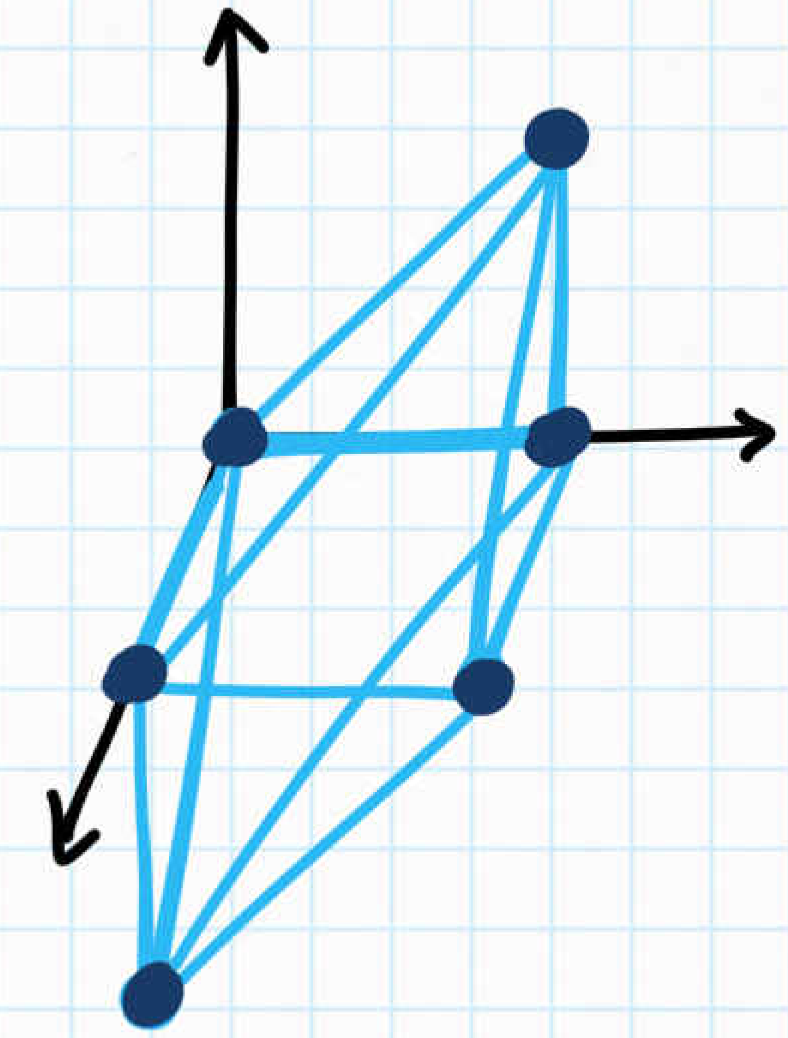


$\mathcal{M}=\mathbb{C}$

* QUIVER FOR M2-BRANES PROBING $C(Q^{1,1,1})$ toric CY_4



$$W = \mathrm{Tr}\left(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1\right) + $$
$$+ \; p_1 B_1 q_1 + p_2 B_2 q_2 \; .$$



[Benini, Closset, SC '09 ; Jafferis '09]

For $\underline{N=1}$, $B = \hat{m} = 0$:

$$H(t,u,v,z) = \sum_{m\in\mathbb{Z}}^{m_1=m_2=m} z^m t^{\frac{1}{2}|m|} \oint \frac{dx}{2\pi i x}\, x^{-|m|}\, \mathrm{PE}\left[t^{\frac{1}{2}} x\left(u+\frac{1}{u}\right) + t^{\frac{1}{2}} \frac{1}{x}\left(v + \frac{1}{v}\right)\right] =$$

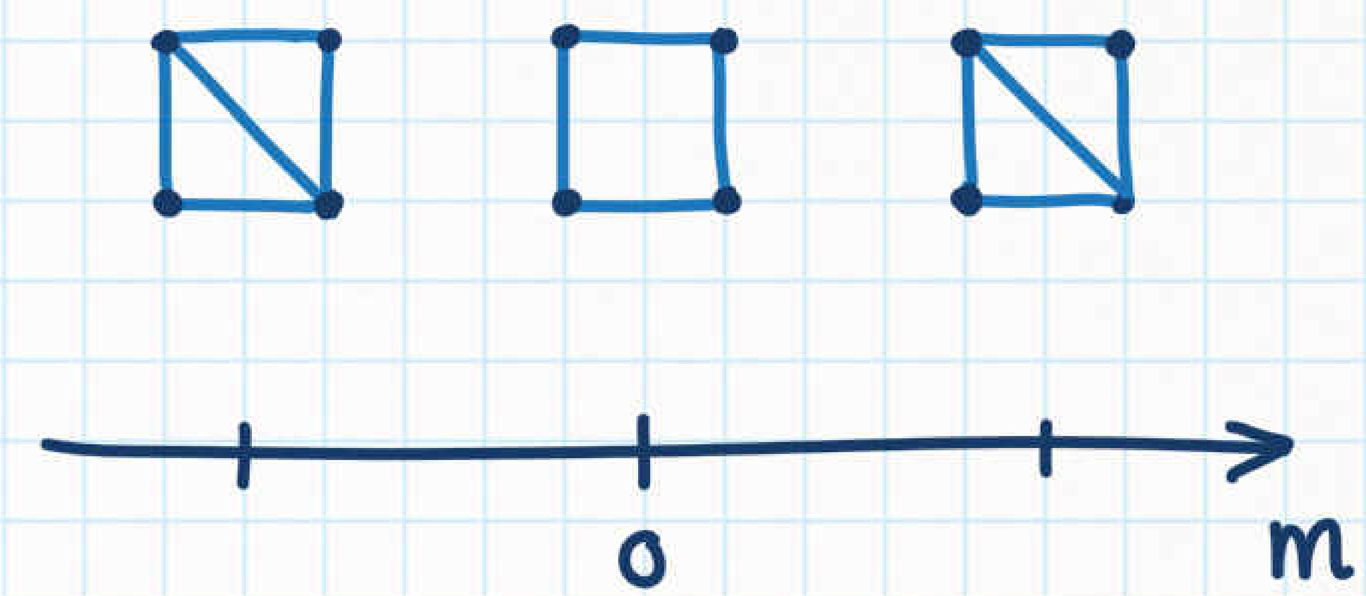
$k_1^{\mathrm{eff}} = -k_2^{\mathrm{eff}} = \mathrm{sign}(m)$

$$= \sum_{m\in\mathbb{Z}} z^m t^{\frac{1}{2}|m|} \cdot g_1\left(t^{\frac{1}{2}}, u, v; |m|\right) = \sum_{n=0}^{\infty} [n;n;n]_{\alpha,\beta,\gamma}\, t^n$$

$u = \alpha$
$v = \gamma\beta$
$z = \gamma/\beta$

$SU(2)^3 \times U(1)_R$

[SC, Mekareeya, Zaffaroni
to appear]



- $N>1$: $\mathcal{M} = \mathrm{Sym}^N C(Q^{1,1,1})$.
- $B \neq 0$, $\hat{m} \neq 0$: resolutions of $C(Q^{1,1,1})$.

The HS formalism can be applied to a large class of M2-brane theories probing CY_4 cones.

OUTLOOK

I presented a general formalism to count gauge invariant chiral operators that parametrize moduli spaces of vacua of 3d $\mathcal{N} \geq 2$ gauge theories.

This simplifies the study of the chiral ring and in favorable cases determines it completely.

SOME OPEN PROBLEMS:

- Explicit chiral ring relations (like [Bullimore, Dimofte, Gaiotto '15] for $\mathcal{N}=4$)
- Mathematical definition (like [Nakajima '15; Braverman, Finkelberg, N. '16] for $\mathcal{N}=4$)
- Path integral derivation as an index (like [Razamat, Willett '14; Closset, Kim '16] for $\mathcal{N}=4$)
  (if $\mathcal{F}$ is a complete intersection)

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THANK YOU FOR YOUR ATTENTION!