(2,0) to 2 M2's

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Introduction/Motivation

In string theory the various Dp-branes are all related to each other in a more or less manifest way by T-duality.

- Their lagrangians all come from reduction of 10D super-Yang-Mills
- Can map between them by constructing periodic arrays [Taylor]
- Of course their quantum properties depend dramatically on the value of p.
- S-dualities are far from manifest

In M-theory S-dualities are more or less manifest (if Lorentz transformations and/or diffeomorphisms are manifest).

There is a T-duality on $\mathbb{T}^3$ inherited from String theory but without a microscopic formulation it seems rather peculiar:

e.g.

$$ds_8^2 + R_1^2 d\theta_1^2 + R_2^2 d\theta_2^2 + R_3^2 d\theta_3^2 \cong \Delta^2 ds_8^2 + \frac{R_2^2}{\Delta^4} d\theta_1^2 + \frac{R_1^2}{\Delta^4} d\theta_2^2 + \frac{R_3^2}{\Delta^4} d\theta_3^2$$

$$\Delta = (R_1 R_2 R_3)^{1/3} / l_p$$

M5-branes and M2-branes are mapped to each other by such a T-duality.

But an M5-brane with momentum is mapped to M2-branes wrapped around a transverse $\mathbb{T}^2$

$$\begin{array}{lcccccc} M5: & 0 & 1 & 2 & 3 & 4 & 5 \\ p: & & & & & & \times \end{array} \quad \overset{T_{345}}{\Longleftrightarrow} \quad \begin{array}{lccccc} M2: & 0 & 1 & 2 & & \\ M2: & 0 & & & 3 & 4 \end{array}$$

Hard to reconcile with a traditional lagrangian picture.

- Might require some M-theory version of the Nahm transform.

Since the lagrangians for M2-branes are known one might have thought that we could use T-duality to construct some sort of lagrangian for M5's.

An attempt along these lines was made in [Jeon, NL, Richmond] which, starting from ABJM, led to a non-manifest Lorentz invariant formulation of 5D Super-Yang-Mills as a worldvolume theory for the M5-brane.

- The origin of the problem is that translational symmetries are not manifest in ABJM lagrangian so the dynamics of periodic arrays are complicated.

But still one might hope to find some structure that encodes the dynamics of both M5-branes and M2-branes in the way that 10D super-Yang-Mills encodes the lagrangians of Dp-branes for all p.

In this talk we will present a representation of the $(2,0)$ super-algebra based on a set of fields which live in a 3-algebra along with a non-dynamical abelian three-form.

- The resulting dynamics give the $\mathcal{N}=8$ M2-brane lagrangian as well as various lagrangians that have been conjectured to describe M5-branes (5D SYM, Instanton QM).

A representation of $(2,0)$

The $(2,0)$ superalgebra of is [NL,Papageorgakis] is

$$
\begin{aligned}
\delta X^i &= i\bar{\epsilon}\Gamma^i\Psi \\
\delta Y^\mu &= 0 \\
\delta\Psi &= \Gamma^\mu\Gamma^i D_\mu X^i + \frac{1}{2\cdot 3!}H_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\epsilon - \frac{1}{2}\Gamma_\mu\Gamma^{ij}[Y^\mu, X^i, X^j]\epsilon \\
\delta H_{\mu\nu\rho} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\rho]}\Psi + i\bar{\epsilon}\Gamma^i\Gamma_{\mu\nu\lambda\rho}[Y^\rho, X^i, \Psi] \\
\delta A_\mu(\cdot) &= i\bar{\epsilon}\Gamma_{\mu\nu}[Y^\nu, \Psi, \ \cdot\ ]
\end{aligned}
$$

$$
\Gamma_{012345}\epsilon = \epsilon \qquad \Gamma_{012345}\Psi = -\Psi
$$

which closes on-shell and subject to the constraints:

$$
\begin{aligned}
D_\mu Y^\nu = [Y^\mu, Y^\nu, \ \cdot\ ] = [Y^\mu, D_\mu\cdot, \ \cdot'\ ] = 0 \\
F_{\mu\nu}(\ \cdot\ ) = [Y^\lambda, H_{\mu\nu\lambda}, \ \cdot\ ]
\end{aligned}
$$

Here $[\,\cdot\,,\ \cdot'\,,\ \cdot''\,]$ is a totally anti-symmetric 3-algebra which we require to have a positive definite inner-product $\langle\,\cdot\,,\ \cdot'\,\rangle$ and satisfy the fundamental identity

$$[A,B,[X,Y,Z]] = [[A,B,X],Y,Z]+[X,[A,B,Y],Z]]+[X,Y,[A,B,Z]]$$

There is just one finite dimensional example corresponding to an $su(2)\oplus su(2)$ Lie-algebra with fields in the $(\mathbf{2},\mathbf{2})$ [Gauntlett, Gutowksi][Papadopoulos]

Note that we can reformulate this without a 3-algebra taking Y^μ to be abelian and replacing $[Y^\mu,\ \cdot\,,\ \cdot'\,] \to Y^\mu[\,\cdot\,,\ \cdot'\,]$

In particular Y^μ is non-dynamical.

We generalise this by including a background 3-form $C_{\mu\nu\lambda}$:

$$\begin{aligned}
\delta X^i &= i\bar{\epsilon}\Gamma^i\Psi \\
\delta Y^\mu &= \frac{i}{2}\bar{\epsilon}\Gamma_{\lambda\rho}C^{\mu\lambda\rho}\Psi \\
\delta H_{\mu\nu\lambda} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\lambda]}\Psi + i\bar{\epsilon}\Gamma^i\Gamma_{\mu\nu\lambda\rho}[Y^\rho, X^i, \Psi] \\
&\quad + \frac{i}{2}\bar{\epsilon}(\star C)_{\mu\nu\lambda}\Gamma^{ij}[X^i, X^j, \Psi] + \frac{3i}{4}\bar{\epsilon}\Gamma_{[\mu\nu|\rho\sigma}C^{\rho\sigma}{}_{\lambda]}\Gamma^{ij}[X^i, X^j, \Psi] \\
\delta A_\mu(\cdot) &= i\bar{\epsilon}\Gamma_{\mu\nu}[Y^\nu, \Psi, \ \cdot\ ] + \frac{i}{3!}\bar{\epsilon}C^{\nu\lambda\rho}\Gamma_{\mu\nu\lambda\rho}\Gamma^i[X^i, \Psi, \ \cdot\ ]\ , \\
\delta\Psi &= \Gamma^\mu\Gamma^i D_\mu X^i\epsilon + \frac{1}{2\cdot 3!}H_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\epsilon - \frac{1}{2}\Gamma_\mu\Gamma^{ij}[Y^\mu, X^i, X^j]\epsilon \\
&\quad + \frac{1}{3!\cdot 3!}C_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\Gamma^{ijk}[X^i, X^j, X^k]\epsilon
\end{aligned}$$

So now Y^μ can be dynamical but we must keep the 3-algebra.

A standard (but trust me tedious) calculation shows that this system indeed closes on the following equations of motion

$$0 = \Gamma^\rho D_\rho \Psi + \Gamma_\rho \Gamma^i [Y^\rho, X^i, \Psi] + \frac{i}{2 \cdot 3!} C^{\rho\sigma\tau} \Gamma_{\rho\sigma\tau} \Gamma^{ij} [X^i, X^j, \Psi]$$

$$0 = D^2 X^i + [Y^\mu, X^j, [Y_\mu, X^j, X^i]] + \frac{1}{2 \cdot 3!} C^2 [X^j, X^k, [X^j, X^k, X^i]]$$

$$+ fermions$$

$$0 = D_{[\lambda} H_{\mu\nu\rho]} + \frac{1}{2} (\star C)_{[\mu\nu\lambda} [X^i, X^j, [Y_{\rho]}, X^i, X^j]]$$

$$+ \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho\sigma\tau} [Y^\sigma, X^i, D^\tau X^i] + fermions$$

As well as constraints:

$$
\begin{aligned}
F_{\mu\nu}(\cdot) &= [Y^{\lambda}, H_{\mu\nu\lambda},\ \cdot\ ] - (\star C)_{\mu\nu\lambda}[X^{i}, D^{\lambda}X^{i},\ \cdot\ ] + fermions \\
0 &= D_{\mu}Y^{\nu} - \frac{1}{2}H_{\mu\lambda\rho}C^{\nu\lambda\rho} \\
0 &= [Y^{\mu}, D_{\mu}(\cdot),\ \cdot'\ ] + \frac{1}{3}[D_{\mu}Y^{\mu},\ \cdot\ ,\ \cdot'\ ] \\
0 &= C^{\mu\nu\lambda}D_{\lambda}(\cdot) - [Y^{\mu}, Y^{\nu},\ \cdot\ ] \\
0 &= C \wedge Y \\
0 &= C_{\sigma[\mu\nu}C^{\sigma}{}_{\lambda]\rho}
\end{aligned}
$$

Somewhat unconventional (ugly?) but I hope to show you that the solutions to these constraints encodes the dynamics of the branes in M-theory.

There is a conserved supercurrent:

$$
\begin{aligned}
S^{\mu} = {} & 2\pi i \langle D_\nu X^i, \Gamma^\nu \Gamma^\mu \Gamma^i \Psi \rangle + \frac{2\pi i}{4} \langle H_{\nu\lambda\rho}, \Gamma^{\nu\lambda\rho} \Gamma^\mu \Psi \rangle \\
& - \frac{2\pi i}{2} \langle [Y_\mu, X^i, X^j], \Gamma^\nu \Gamma^\mu \Gamma^{ij} \Psi \rangle \\
& + \frac{2\pi i}{3!^2} C_{\nu\lambda\rho} \langle [X^i, X^j, X^k], \Gamma^{\nu\lambda\rho} \Gamma^\mu \Gamma^{ijk} \Psi \rangle
\end{aligned}
$$

and energy-momentum tensor:

$$
\begin{aligned}
T_{\mu\nu} = {} & 2\pi \langle D_\mu X^i, D_\nu X^i \rangle - \pi \eta_{\mu\nu} \langle D_\lambda X^i, D^\lambda X^i \rangle \\
& - \frac{\pi}{2} \eta_{\mu\nu} \langle [Y_\lambda, X^i, X^j], [Y^\lambda, X^i, X^j] \rangle \\
& + \frac{2\pi}{3!} (C_{\mu\lambda\rho} C_\nu{}^{\lambda\rho} - \frac{1}{6} \eta_{\mu\nu} C^2) \langle [X^i, X^j, X^k], [X^i, X^j, X^k] \rangle \\
& + \frac{\pi}{2} \langle H_{\mu\lambda\rho}, H_\nu{}^{\lambda\rho} \rangle + fermions
\end{aligned}
$$

One can also compute the superalgebra and central charges.

Solving the constraints: Reduction to 2 M5's

Let us recall the case of $C_3 = 0$. Here can fix

$$Y^\mu = V^\mu T^4$$

where T^4 is some generator of the 3-algebra and V^μ a constant vector:

- All fields parallel to T^4 become free - 6D centre of mass (2,0) multiplet
- Remaining fields are acted on by an $su(2)$ gauge algebra and only depend on the the coordinates orthogonal to V^μ.

But there are still some choices:

We can fix $Y^\mu = g^2\delta^\mu_5 T^4$ (spacelike).

The constraints then say that the remaining dynamical fields only depend on $x^0, ..., x^4$ and we recover (4+1) SYM with

$$F_{\mu\nu} = g^2 H_{\mu\nu 5}$$

Claimed to be a complete description of the M5-brane on S^1 of radius $R = g^2/4\pi^2$ [Douglas][NL, Papageorgakis, Schmidt-Sommerfeld]

Alternatively we can set $Y^\mu = g^2\delta^\mu_0$ and find (5+0) E-SYM with $F_{\mu\nu} = g^2 H_{\mu\nu 0}$ [Hull][Hull,Khuri] gives the M5-brane with emergent compact time [Hull,NL]

Alternatively we can set $Y^\mu = g^2\delta^\mu_+$

Here we find

$$F_{\mu\nu} = g^2 H_{\mu\nu+}$$

and self-duality of H leads to self-duality of $F_{\mu\nu}$

- spatial equations can all be solved by the ADHM construction
- dynamics reduced to motion on instanton-moduli space with x^- as 'time' and $p_+ = n/R$ [NL, Richmond]

This reproduces the DLCQ conjecture for the dynamics of M5-brane [Aharony,Berkooz,Seiberg][Aharony, Kachru, Seiberg,Silverstein]

Reduction to 2 M2's

Let us take $C_{345} = l^3$ non-vanishing.

The constraint

$$[Y^\mu, D_\mu \cdot\,, \cdot'\,] + \frac{1}{3}[D_\mu Y^\mu,\, \cdot\,,\, \cdot'\,] = 0$$

suggests setting $\partial_a = 0$, $a = 3, 4, 5$ and $Y^\alpha = 0$, $\alpha = 0, 1, 2$.

In which case the constraint

$$C^{\mu\nu\lambda} D_\lambda(\cdot) - [Y^\mu, Y^\nu,\, \cdot\,] = 0$$

implies

$$A_a(\cdot) = -\frac{1}{2l^3}\varepsilon_{abc}[Y^b, Y^c,\, \cdot\,]$$

From this the remaining constraints can solved leading to

$$H_{abc} = -\frac{1}{l^6}[Y_a, Y_b, Y_c]$$

$$H_{\alpha bc} = -\frac{1}{l^3}\varepsilon_{bcd} D_\alpha Y^d$$

$$H_{\alpha\beta c} = -\frac{1}{l^3}\varepsilon_{\alpha\beta\gamma} D^\gamma Y_c$$

$$H_{\alpha\beta\gamma} = -\frac{1}{3! l^6}\varepsilon_{\alpha\beta\gamma}\varepsilon^{abc}[Y_a, Y_b, Y_c]$$

A similar form was also used in [Ho,Matsuo] using Nambu brackets.

We also find

$$F_{\alpha a}(\cdot) = \frac{1}{l^3}\varepsilon_{abc}[Y^b, D_\alpha Y^c,\ \cdot\ ] + fermions$$

$$F_{ab}(\cdot) = \frac{1}{l^6}[Y^c, [Y_a, Y_b, Y_c],\ \cdot\ ] + fermions$$

Let us now look at the supersymmetry transformations.

$$\delta\Psi = \Gamma^\alpha\Gamma^i D_\alpha X^i\epsilon - \frac{1}{2l^3}\Gamma^{ab}\Gamma_{345}\Gamma^i[Y^a,Y^b,X^i]\epsilon - \frac{1}{3!l^6}\Gamma_{abc}[Y^a,Y^b,Y^c]\epsilon$$
$$- \frac{1}{2}\Gamma^a\Gamma^{ij}[Y^a,X^i,X^j]\epsilon + \frac{1}{3!^2}\Gamma_{345}\Gamma^{ijk}[X^i,X^j,X^k]\epsilon$$

For the bosons we find

$$\delta X^i = i\bar\epsilon\Gamma^i\Psi$$
$$\delta Y^\alpha = il^3\alpha\bar\epsilon\Gamma_\alpha\Gamma^{345}\Psi$$
$$\delta Y^a = il^3\bar\epsilon\Gamma^a\Gamma_{345}\Psi$$
$$\delta A_\alpha(\cdot) = i\bar\epsilon\Gamma_\alpha\Gamma^b[Y^b,\Psi,\ \cdot\ ] - il^3\bar\epsilon\Gamma_\alpha\Gamma_{345}\Gamma^i[X^i,\Psi,\ \cdot\ ]$$

Let us write $X^a = l^{-3/2} Y^a$ and introduce new spinors

$$\epsilon' = \Omega\epsilon \qquad \Psi' = l^2\Omega\Psi \qquad \Omega = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\Gamma_{345}$$

$$\Rightarrow \Gamma_{012}\epsilon' = \epsilon' \qquad \Gamma_{012}\Psi' = -\Psi'$$

Then everything reduces to ($I = 3, 4, 5, ..., 10$)

$$\begin{aligned} \delta\Psi' &= \Gamma^\alpha\Gamma^I D_\alpha X^I \epsilon' - \frac{1}{3!}\Gamma^{IJK}[X^I, X^J, X^K]\epsilon' \\ \delta X^I &= i\bar{\epsilon}'\Gamma^I\Psi' \\ \delta A_\alpha(\cdot) &= i\bar{\epsilon}'\Gamma_\alpha\Gamma^I[X^I, \Psi', \cdot] \end{aligned}$$

This is the maximally supersymmetric M2-brane theory (equations of motion work out too).

Describes two M2-branes in $\mathbb{R}^8$, $\mathbb{R}^8/\mathbb{Z}_2$ [NL, Tong][Distler, Muhki, Papageorgakis, van Raamsdonk] [Bashkirov,Kapustin] [NL, Papageorgakis]

We can also take a “timelike” $C_{045} = l^3$.

Gives a Euclidean theory on x^1, x^2, x^3 with no time dependence.

This leads to a maximally supersymmetric Euclidean M2-brane theory with $SO(2,6)$ R-symmetry.

- Similar in structure to the maximally supersymmetric case but with some funny signs
- Worldvolume theory of an E2-brane in $5+6$ dimensional M-theory [Hull][Hull,Khuri]

Momentum Modes

Let us return to the question of T-duality with momentum modes

$$\begin{array}{lcccccc} M5: & 0 & 1 & 2 & 3 & 4 & 5 \\ p: & & & & & & \times \end{array} \quad \overset{T_{345}}{\Longleftrightarrow} \quad \begin{array}{lccccc} M2: & 0 & 1 & 2 & & \\ M2: & 0 & & & 3 & 4 \end{array}$$

Here we find

$$\begin{aligned} P_5 &= \int d^5x T_{05} \\ &= \frac{\pi}{2} \int d^5x \langle H_{0\lambda\rho}, H_5{}^{\lambda\rho} \rangle \\ &= -\frac{\pi}{4} \varepsilon^{05\lambda\rho\mu\nu} \int d^5x \langle H_{5\mu\nu}, H_{5\lambda\rho} \rangle \end{aligned}$$

Where we have restricted attention to $SO(5)$ R-symmetry singlet states

First recall the M5-brane case: $C_3 = 0$

Here we can fix $Y^\mu = g^2 \delta_5^\mu T^4$ then $H_{5\mu\nu} = g^{-2} F_{\mu\nu}$ and hence

$$P_5 = -\left(\frac{4\pi R}{g^2}\right)^2 \cdot \frac{1}{R} \cdot \frac{1}{8\pi^2} \int d^4x \langle F \wedge F \rangle$$

leading to a KK spectrum

$$P_5 = n/R$$

if $R = g^2/4\pi$.

So we have compact circle (even though we didn't ask for it).

Next consider the M2-brane case: $C_3 = l^3 dx^3 \wedge dx^4 \wedge dx^5$

Here we found ($\alpha, \beta = 0, 1, 2,\ a, b = 3, 4, 5$)

$$H_{5\alpha\beta} = -\frac{1}{l^3}\varepsilon_{\alpha\beta\gamma}D^\gamma Y_5$$
$$H_{5\alpha b} = -\frac{1}{l^3}\varepsilon_{bc5}D_\alpha Y^c$$
$$H_{5ab} = -\frac{1}{l^6}[Y_5, Y_b, Y_c]$$

and hence

$$P_5 = -\frac{16\pi^4 R^3}{l^9}\int d^2x \langle [Y^3, Y^4, Y^5], D_0 Y^5\rangle + l^3\varepsilon^{0\alpha\beta}\langle D_\alpha Y^3, D_\beta Y^4\rangle$$

And the last term indeed counts M2-brane wrapping around x^3, x^4.

$P_5 = n/R$ if Y^a parameterize a 3-torus with radii $R = l/2\pi$.

Conclusions/Comments

In this talk we gave a classical system of equations of motion that furnish a representation of the 6D (2,0) superalgebra.

- Motivated by the belief in some kind of M-theory T-duality.
- Generalised a previous construction that leads to various theories associated to the M5-brane (5D SYM, 5D E-SYM, QM on instanton moduli space)
- Turning on the abelian 3-form leads to the maximally supersymmetric theory of 2 M2-branes.
- More generally it is interesting to seek field theories whose symmetries correspond to those of extended objects embedded in eleven-dimensions, not just ten.

What is the role of the 3-algebra? Only one finite-dimensional positive definite case describing two M2's or 2 M5's branes.

- Maybe, as with M2-branes, this allows for more manifest symmetries, and we can hope that the structure persists more generally but less cleanly.
- Need to generalize the construction to ABJM-type theories - Should lead to a curious formulation of the M5-brane.
- Could also consider Lorentzian 3-algebras - leads to Dp-branes when $C_3 = 0$ - [Honma, Ogawa Shiba],[Kawamoto, Takimmi, Tomino]
- Relation to Higher gauge theory [Saemann,...,Wolf,...]

It is tempting to speculate on the interpretation of C_3.

- looks like a background C-field and some kind of Myers effect
- due to conformal symmetry it is either “on” or “off”.
- similar to the construction of an abelian M5-brane from M2-branes with a Nambu bracket [Ho, Matsuo],[Ho,Imamura, Shiba],[Bandos, Townsend]
- It would be interesting to look at other choices of C_3 (“null”, self-dual,...)

Some more random comments among friends:

- It is important to note that M-theory is the strong coupling limit = UV (M5) or IR (M2) of the gauge theory
- One must be careful about T-duality since there are now explicit scales arising from the radii.
  - For the M5-branes radii go to infinity
  - For the M2-branes radii go to zero
- Since $C_{\mu\nu\lambda}$ has dimensions of (length)^3 it effectively “runs” to infinity in the UV for the M5-brane unless it vanishes. This might explain why it is either “off” (M5) or “on” (M2).