Two dimensional $\mathcal{N} = (0, 2)$ theories and Calabi-Yau 4-algebras

Richard Eager
Universität Heidelberg

Supersymmetric Theories, Dualities and Deformations
Albert Einstein Center
Institute for Theoretical Physics
Bern - Switzerland

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Outline:

Following orders:

- Supersymmetric Theories
- Dualities
- Deformations

As the title indicates, the supersymmetric theories will be 2d $\mathcal{N} = (0,2)$ theories obtained from D1-branes at CY-4 singularities.

Motivation

After considering,

- D3-branes at CY singularities,
- M2-branes at CY singularities,

the next logical step would be

- D1-branes at CY singularities.

But that last slide wasn't so motivating . . .

More seriously, considering branes at singularities has taught us much about holography.

"All happy families are alike; each unhappy family is unhappy in its own way."

Each brane has its own idiosyncrasies.

In particular 3D theories taught us much about monopole operators and the Coulomb branch.

2D $\mathcal{N} = (0, 2)$ theories have a reputation for being difficult, but the ones arising from D1 branes at singularities seem to be well under control.

Renewed interest in $(0, 2)$

- Elliptic genus from supersymmetric localization [F. Benini, R. E. , K. Hori, Y. Tachikawa] [A. Gadde, S. Gukov]
- Brane brick models [S. Franco, S. Lee, R. Seong]
- F-theory constructions [S. Schafer-Nameki, T. Weigand]

$\mathcal{N} = (8, 8)$ SYM in 2D

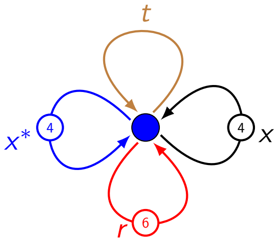


Figure: dg quiver corresponding to $\mathbb{C}^4$

$\mathcal{N} = (0,2)$ superspace

We introduce $\mathcal{N} = (0,2)$ superspace to construct manifestly supersymmetric Lagrangrangians. The bosonic coordinates are $y^\alpha, \alpha = 1,2$ and fermionic coordinates are $\theta^+, \overline{\theta}^+$. The two supersymmetry generators are

$$\mathcal{Q}_+ = \frac{\partial}{\partial \theta^+} + i\overline{\theta}\left(\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^1}\right) \quad (1.1)$$

$$\overline{\mathcal{Q}}_+ = \frac{\partial}{\partial \overline{\theta}^+} - i\theta\left(\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^1}\right). \quad (1.2)$$

The supersymmetry generators commute with the super derivatives

$$D_+ = \frac{\partial}{\partial \theta^+} - i\overline{\theta}\left(\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^1}\right) \quad (1.3)$$

$$\overline{D}_+ = \frac{\partial}{\partial \overline{\theta}^+} + i\theta\left(\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^1}\right). \quad (1.4)$$

$\mathcal{N} = (0, 2)$ theories

There are two types of matter multiplets which we now introduce. The first matter field we consider is the $(0, 2)$ chiral multiplet Φ which obeys

$$\overline{\mathcal{D}}_+ \Phi = 0$$

and has components

$$\Phi = \phi + \sqrt{2}\theta^+ \psi_+ - i\theta^+ \overline{\theta}^+ (D_0 + D_1)\phi.$$

The Fermi multiplet Λ_- satisfies

$$\overline{\mathcal{D}}_+ \Lambda_- = \sqrt{2} E$$

where E is a superfield obeying

$$\overline{\mathcal{D}}_+ E = 0.$$

Interactions

We will only need to consider the case where $E = E(\Phi_i)$ is a holomorphic function of some chiral superfields Φ_i. In addition to the E-interaction, we can introduce a J-interaction using the $(0,2)$ analog of the superpotential:

$$L_J = \int d^2y d\theta^+ \, \Lambda_{-,a} J^a(\Phi_i)|_{\overline{\theta}^+=0}.$$

For this term to be supersymmetric, the following constraint must be satisfied:

$$\sum_a E_a(\Phi_i) J^a(\Phi_i) = 0.$$

The E and J-interactions can be exchanged be replacing the Fermi multiplet Λ_- with its conjugate $\overline{\Lambda}_-$.

Calabi-Yau algebras in physics

- Quivers have been intensely studied in the context of D-branes
- A particular feature of D3-branes at a Calabi-Yau threefold singularity is that all of the relations come from a single function known as the superpotential.
- A form of Serre duality results in the quiver with potential having a self-dual resolution as a non-commutative algebra [Berenstein-Douglas].
- This led to the notion of a 3-Calabi Yau algebra in [van den Bergh] and to Calabi-Yau *n*-algebras in Ginzburg.
- We introduce the structure of Calabi-Yau 4-algebras following [Lam].

Calabi-Yau 4 algebras from geometry

Given a (Gorenstein) CY4 singularity, we can always construct a CY-4 algebra from a NCCR. Fortunately these have been well studied by mathematicians. Physically, we obtain a NCCR from a collection of branes satisfiying the "Grade-Restriction Rule" [abelian – Hori-Herbst-Page '08, non-abelian – R.E.-Hori-Knapp-Romo '18].

This can be used to rederive the theory of brane tilings [RE arXiv:1003.2862] and its higher dimensional generalization needed for CY 4 singularities. Also one of the first hints of cluster algebras and integrable systems.

Calabi-Yau 4-algebras from quivers

A quiver Q can be described as a collection of vertices Q_0, arrows Q_1 and maps $h : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ called the "head" and "tail" of an arrow. A Calabi-Yau 4-algebra consists of the following data.

1. A quiver $Q = (Q_0, Q_1, h : Q_1 \to Q_0, t : Q_1 \to Q_0)$.
2. A map $A : Q_1 \to \mathbb{C}Q$ such that $A_r := A(r) \in e_{h(r)}\mathbb{C}Qe_{t(r)}$.
3. A symmetric function $q : Q_1 \times Q_1 \to \mathbb{C}$ such that
    1. $q(r,s) = 0$ unless r and s have the same underlying edge are oppositely oriented. That is to say, $h(r) = t(s)$ and $h(s) = t(r)$.
    2. q is nondegenerate – meaning that the matrix $q(r,s)$ is invertible.
    3. $\sum_{r,s \in Q_1} q(r,s) A_r A_s = 0 \mod [\mathbb{C}Q, \mathbb{C}Q]$.

Remarkably, this can be precisely translated into the structure of a $(0,2)$ theory.

$(0,2)$ theories from Calabi-Yau 4 algebras

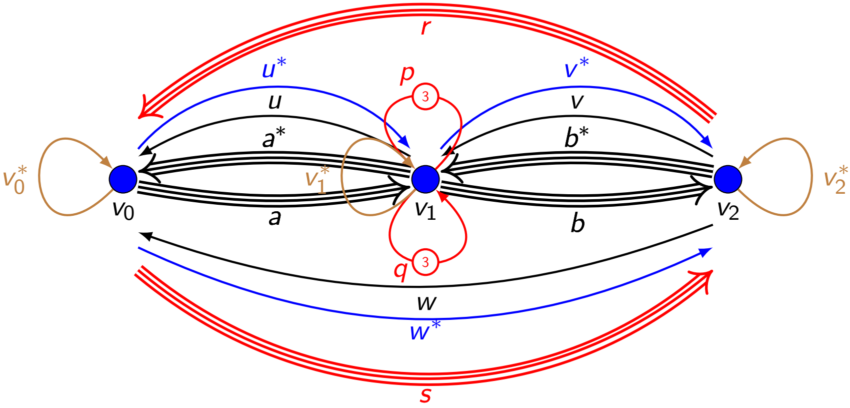
Essentially, the map $A : Q_1 \to \mathbb{C}Q$ such that $A_r := A(r) \in e_{h(r)}\mathbb{C}Qe_{t(r)}$ specifies both the J and E terms. Furthermore the condition

$$\sum_{r,s\in Q_1} q(r,s)A_r A_s = 0 \mod [\mathbb{C}Q, \mathbb{C}Q]$$

is precisely the constraint that

$$\sum_a E_a(\Phi_i)J^a(\Phi_i) = 0.$$

$\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathbb{P}^2$



$\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathbb{P}^2$

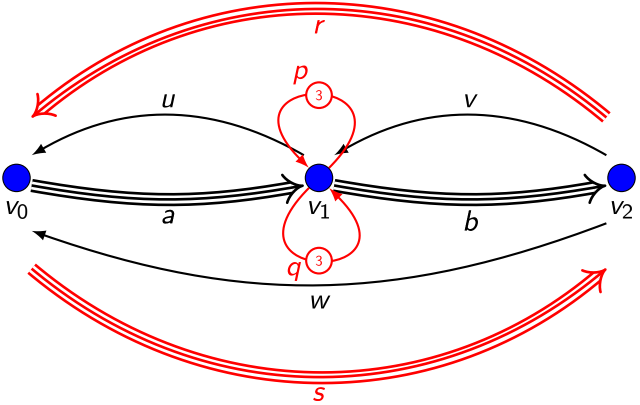


Figure: Underlying quiver corresponding to $\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathbb{P}^2$

Dualities

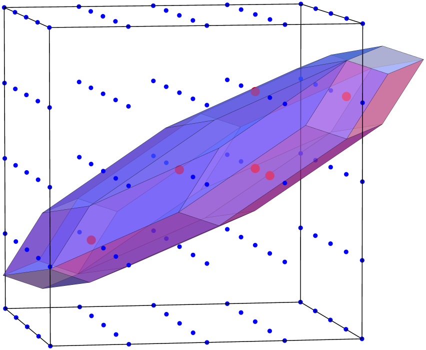


Figure: Translations $\leftrightarrow$ Dualities, From [R.E., S. Franco arXiv:1112.1132]

Previous applications of Calabi-Yau algebras

- Equivalence of A-maximization and volume minimization [Butti-Zaffaroni '05] [R.E. '10]
- Matching of superconformal index
- D3 branes – [R.E., J. Schmude, Y. Tachikawa] arXiv:1207.0573
- M2 branes – [R.E., J. Schmude arXiv:1305.3547]
- Matching of protected operators [R.E. arXiv:1510.04078]

Many of these results generalize to 2d theories!

Motivation from Holography

Gauge Theory

$\mathbb{R}^{3,1} \times X_6$
N $D3$ branes
X_6 Calabi-Yau 6-manifold

Gravity Theory

$AdS_5 \times L_5$
N units of RR-flux
L_5 Sasaki-Einstein 5-manifold

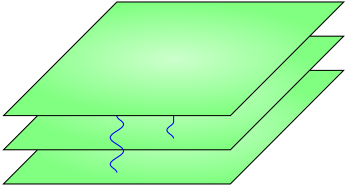


Figure: N D3-branes

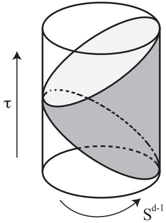


Figure: AdS Space-Time

Goal: Match Closed String States in the large-N limit

Gauge Theory

$\mathbb{R}^{3,1} \times X_6$

Closed strings:

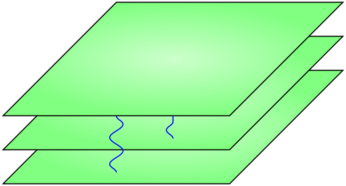
$HC_\bullet(\mathbb{C}Q/\partial W)$

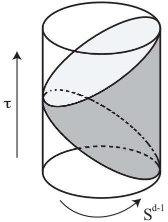
Gravity Theory

$AdS_5 \times L_5$

Closed strings:

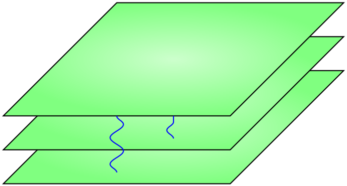
$HP_\bullet(X, \pi = 0)$





$\mathcal{N} = 4$ SYM

N D3 branes filling $\mathbb{R}^{1,3}$ in $\mathbb{R}^{1,3} \times \mathbb{C}^3$.



$\mathcal{N} = 4$ SYM has superpotential

$$W = \mathrm{Tr}\left(XYZ - XZY\right)$$

where X, Y, Z are adjoint-valued chiral superfields.

Superpotential algebra

$$\mathcal{A} = \mathbb{C}\langle x, y, z\rangle/(xy - yx, yz - zy, zx - xz) \cong \mathbb{C}[x, y, z]$$

Protected operators in $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM has three adjoint chiral scalar superfields Φ^1, Φ^2, Φ^3. Their interactions are described by the superpotential

$$W = \operatorname{Tr} \Phi^1 \left[\Phi^2, \Phi^3\right].$$

Consider an operator of the form

$$\mathcal{O} = T^{z_1 z_2 \dots z_k} = \operatorname{Tr} \Phi^{z_1} \Phi^{z_2} \dots \Phi^{z_k}.$$

If $T^{z_1 z_2 \cdots z_k}$ is symmetric in its indices, then the operator is in a short representation of the superconformal algebra. If $T^{z_1 z_2 \cdots z_k}$ is not symmetric, then the operator is a descendant, because the commutators $[\Phi^{z_i}, \Phi^{z_j}]$ are derivatives of the superpotential W [Witten '98].

Matching protected operators in $\mathcal{N} = 4$ SYM

Under the AdS/CFT dictionary, a scalar excitation Φ in AdS obeying

$$(\Box_{AdS_5} - m^2)\Phi = 0$$

with asymptotics $\rho^{-\Delta}$ near the boundary of AdS ($\rho \to \infty$) is dual to an operator of scaling dimension

$$m^2 = \Delta(\Delta - d) \to \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} = m^2}$$

Matching protected operators in $\mathcal{N} = 4$ SYM

The operator

$$\mathcal{O} = \mathrm{Tr}\, \Phi^{z_1} \Phi^{z_2} \dots \Phi^{z_k}$$

has conformal dimension k and is dual to a supergravity state of spin zero and mass

$$m^2 = k(k-4).$$

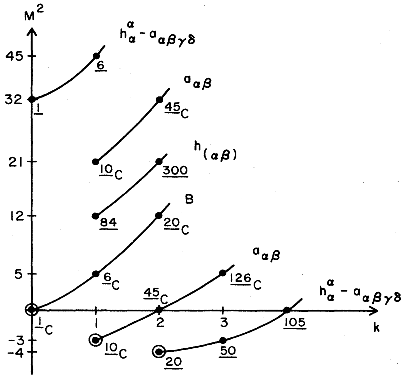


FIG. 2. Mass spectrum of scalars.

Figure: From Kim-Romans-van

Goal: Test AdS/CFT by small deformations

$\mathcal{N} = 4$ SYM has superpotential

$$W = \mathrm{Tr}\left(XYZ - XZY\right).$$

What happens when we deform it by giving a mass to one of the scalars

$$W = \mathrm{Tr}\left(XYZ - XZY + mZ^2\right)$$

or deform the coupling constants?

$$W = \mathrm{Tr}\left(qXYZ - q^{-1}XZY\right)$$

Can we still match the spectrum of protected operators?

Operators in $\mathcal{N} = 4$ Super Yang-Mills

For $X = \mathbb{C}^3, L^5 = S^5$. The corresponding gauge theory is $\mathcal{N} = 4$ SYM, whose superpotential algebra is

$$\mathcal{A} = \mathbb{C}\langle x, y, z\rangle/(xy - yx, yz - zy, zx - xz) \cong \mathbb{C}[x, y, z]$$

	1	t^2	t^4	t^6	t^8	t^{10}	t^{12}	...
HC_0	1	3	6	10	15	21	28	...
HC_1	0	0	3	8	15	24	35	...
HC_2	0	0	0	1	3	6	10	...
$\mathcal{I}(t)$	1	3	3	3	3	3	3	...

Table: Cyclic homology group dimensions for $\mathcal{N} = 4$ SYM

Elements $\mathcal{O} \in HC_0(\mathcal{A}) = \mathcal{A}/[\mathcal{A}, \mathcal{A}]$ are of the form

$$\mathcal{O} = \operatorname{Tr} x^i y^j z^k, \qquad i, j, k \in \mathbb{N}_{\geq 0}$$

The β-deformation

The β-deformation of $\mathcal{N} = 4$ super Yang-Mills theory is a quiver gauge theory with potential $W = qxyz - q^{-1}xzy$ where $q = e^{i\beta}$. The F-term relations are

$$xy = q^{-2}yx$$
$$yz = q^{-2}zy$$
$$zx = q^{-2}xz$$

The cyclic homology groups were computed by Nuss and Van den Bergh.

Chiral Primaries in the β-deformation

Consider an operator $\mathcal{O} = \mathrm{Tr}\, l_1 l_2 \dots l_n$, where l_i is one of the letters x, y, or z. Suppose that l_1 is an x. The F-term conditions imply that

$$\mathcal{O} = \mathrm{Tr}\, l_1 l_2 \dots l_{n-1} l_n = q^{2(|z|-|y|)}\, \mathrm{Tr}\, l_n l_1 l_2 \dots l_{n-1},$$

where $|x|, |y|$, and $|z|$ are the total number of x's, y's, and z's in the operator $\mathcal{O}$. Thus the single-trace chiral primaries have charges $(k, 0, 0), (0, k, 0), (0, k, 0), (k, k, k)$ [D. Berenstein, V. Jejjala, R. G. Leigh]. [1] For q a k-th root of unity, the cyclic homology groups jump.

[1] For $G = SU(N)$ there are additional chiral primaries $\mathrm{Tr}\, xy, \mathrm{Tr}\, xz$ and $\mathrm{Tr}\, yz$. This agrees with the perturbative one-loop spectrum of chiral operators found in [D. Z. Freedman, U. Gursoy].

Operators in the β-deformation

Cyclic homology gives a prediction for the spectrum of protected operators in the β-deformation. The corresponding gravity solution was found by Lunin and Maldacena.

	1	t^2	t^4	t^6	t^8	t^{10}	t^{12}	...
HC_0	1	3	3	4	3	3	4	...
HC_1	0	0	0	2	0	0	2	...
HC_2	0	0	0	1	0	0	1	...
$\mathcal{I}(t)$	1	3	3	3	3	3	3	...

Table: Cyclic homology group dimensions for the β-deformation

Massive Deformation

After adding a mass deformation $\Delta\mathcal{L} = \operatorname{Tr} mz^2$, to $\mathcal{N} = 4$ super Yang-Mills, the superpotential is $W = xyz - xzy + mz^2$. Since z is massive, it can be integrated out of the Lagrangian using its equations of motion. The result is $W = \frac{1}{m}[x, y]^2$. Both superpotential algebras are Morita equivalent and have the same $\mathcal{Q}$ cohomology groups. The F-term relations are

$$\begin{aligned}
[x, y] &= z \\
[x, z] &= 0 \\
[y, z] &= 0.
\end{aligned}$$

Massive Deformation II

The Q-cohomology for the massive deformation is

	1	$t^{3/2}$	t^3	$t^{9/2}$	t^6	$t^{15/2}$	t^9	...
HC_0	1	2	3	4	5	6	7	...
HC_1	0	0	0	2	3	4	5	...
HC_2	0	0	0	0	1	0	1	...
$\mathcal{I}(t)$	1	2	3	2	3	2	3	...

Table: Cyclic homology group dimensions for the massive deformation

We will compare these protected operators to the short representations in the KK-spectrum of the exact SUGRA solution found by Pilch and Warner.

Pilch-Warner Solution

A new critical point of $\mathcal{N} = 8$ gauge supergravity on AdS_5 was discovered by Khavaev-Pilch-Warner. Pilch and Warner found the full type IIB supergravity solution.

$$ds_{10}^2 = \Delta^{-1} ds_{AdS_5}^2 + L^2 \Delta^1 ds_5^2(\rho, \chi)$$

$$ds_5^2(\rho, \chi) = (dx^I Q_{IJ}^{-1} dx^J) + \frac{\sinh^2 \chi}{\xi^2} (x^I J_{IJ} dx^J)^2$$

ρ and χ are critical points of the supergravity potential. For the Pilch-Warner critical point $\rho = 2^{1/6}, \chi = \frac{1}{2} \log 3$. The warp-factor is

$$\Delta = \Omega^{-2}$$

where $\Omega^2 = \xi \cosh \chi$.

Glueball spectrum

The KK-spectrum of glue balls is found by finding solutions of the warped-Laplacian

$$\mathcal{L} \equiv \frac{\Delta^{-1}}{\sqrt{-g_5}} \partial_\alpha \left(\sqrt{-g_5} \Delta^{-1} g^{\alpha\beta} \partial_\beta \right)$$

The short KK multiplets of the graviton exactly match the prediction from the second cyclic homology group.

Further applications of cyclic homology

For CY-3 algebras

$$HC_j(\mathcal{A}) = 0 \text{ for } j > 2$$

This corresponds to the AdS dual theory having no particles of spin higher than 2.

$$HC_2(\mathcal{A}) = Z(\mathcal{A})$$

So the KK-spectrum of gravitons can be computed from the center of the superpotential algebra. For the Pilch-Warner solution, this has been checked explicitly.

Final Remarks

2d gauge theories

Found a precise relationship between geometry and physics. Many exciting directions!

Deformations

We have shown how to compare the protected fields on both sides of the AdS/CFT correspondence at large-N.

- Further extension to finite N is possible, although the cyclic homology groups become much harder to compute.

Thank you for listening!