Elliptic algebras and large-N supersymmetric gauge theories

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1510.00972 1601.08238 with A. Sciarappa and in progress with S. Gukov

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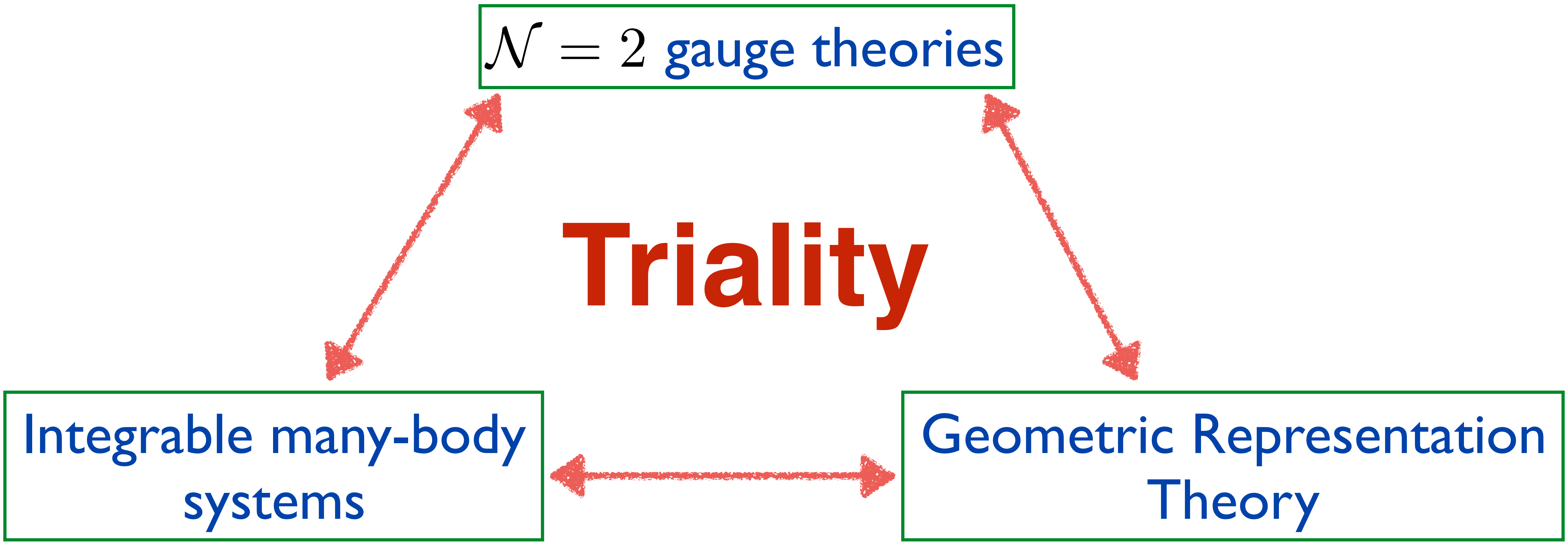
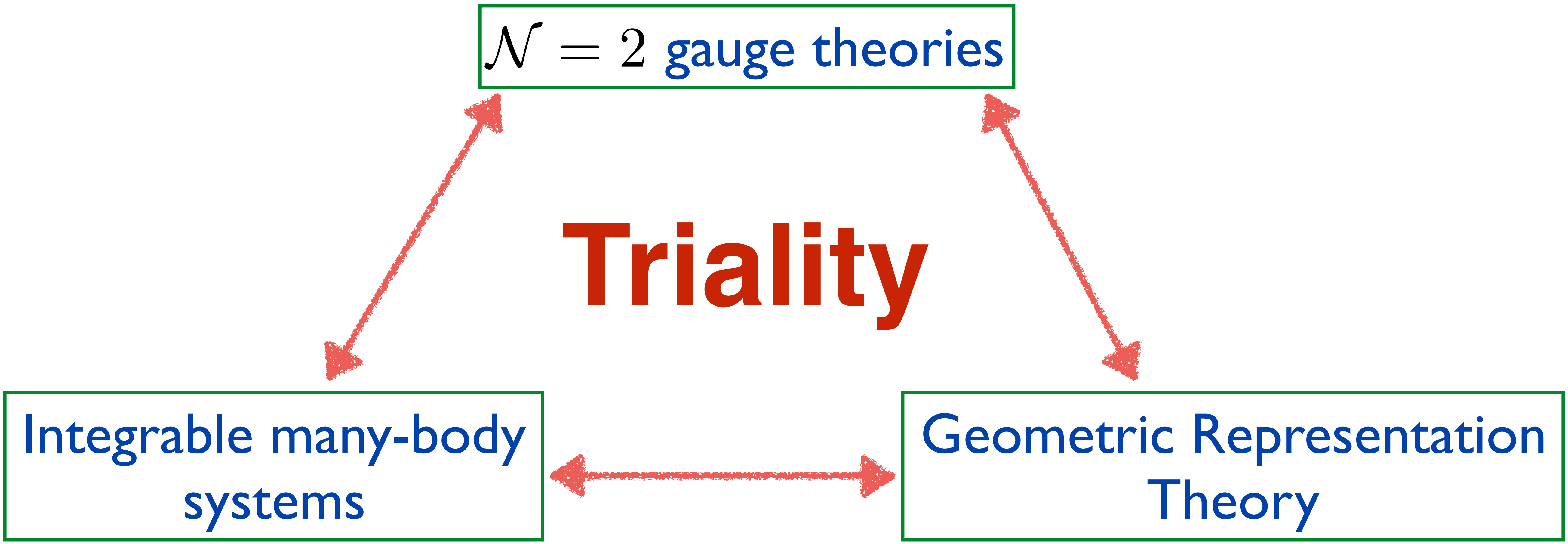
Large-N Gauge Theories

Gauge theories are known to have effective descriptions when the number of colors is large $U(N)$ $N \to \infty$

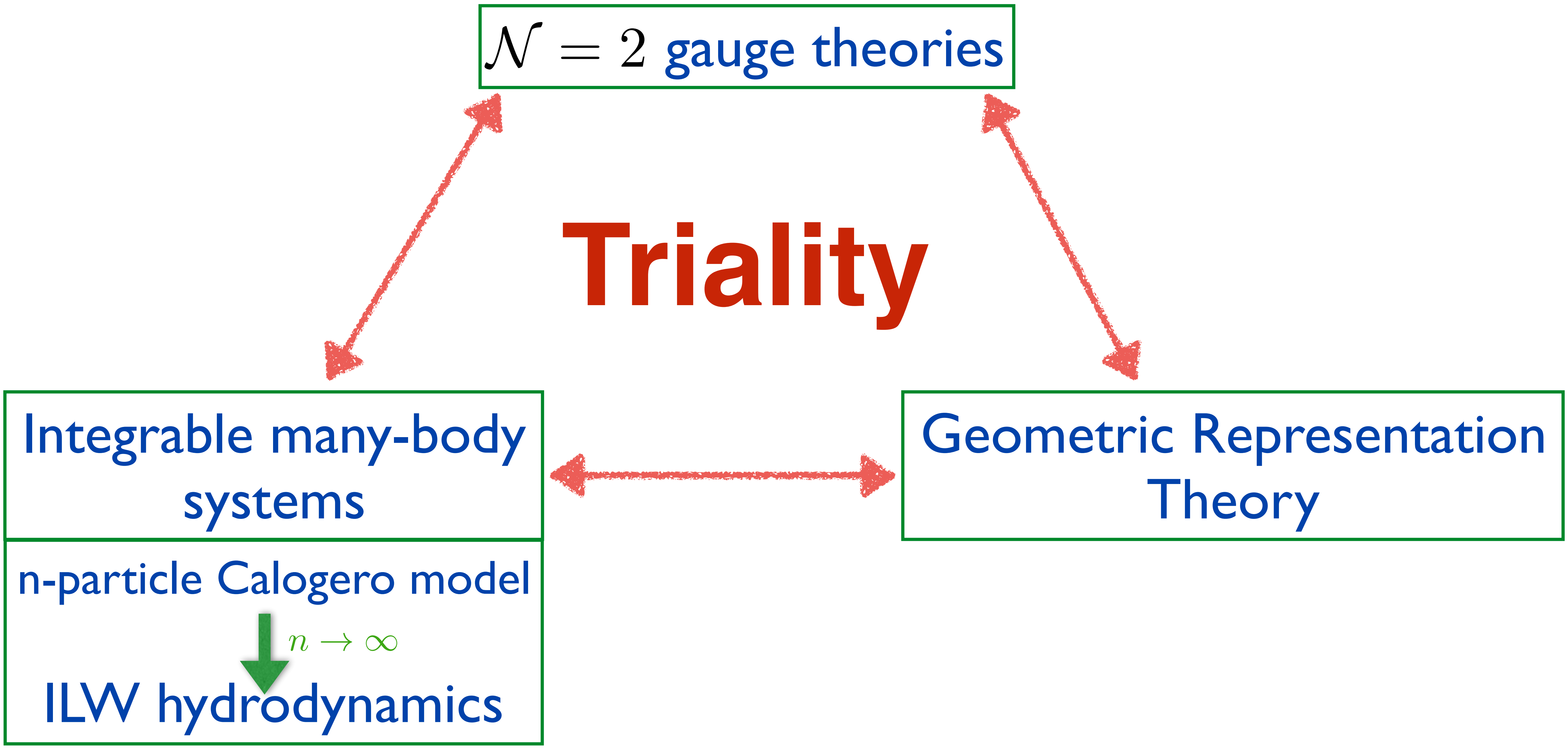
For supersymmetric gauge theories we expect to compute the effective large-N theory exactly

There are plenty of examples in the literature


$\mathcal{N} = 2$ gauge theories
Triality
Integrable many-body systems
Geometric Representation Theory



Large-n limits are manifest in each description!



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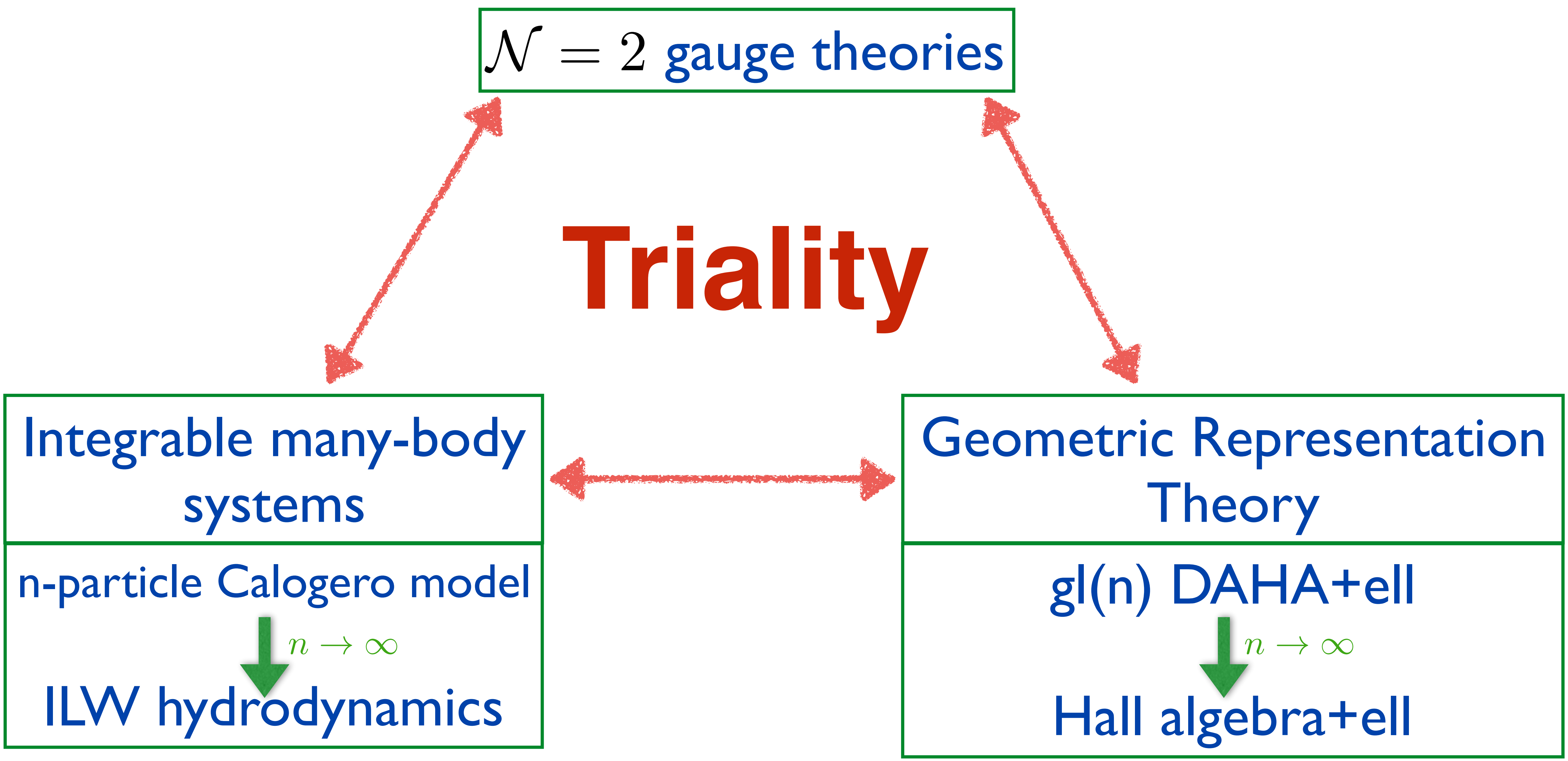

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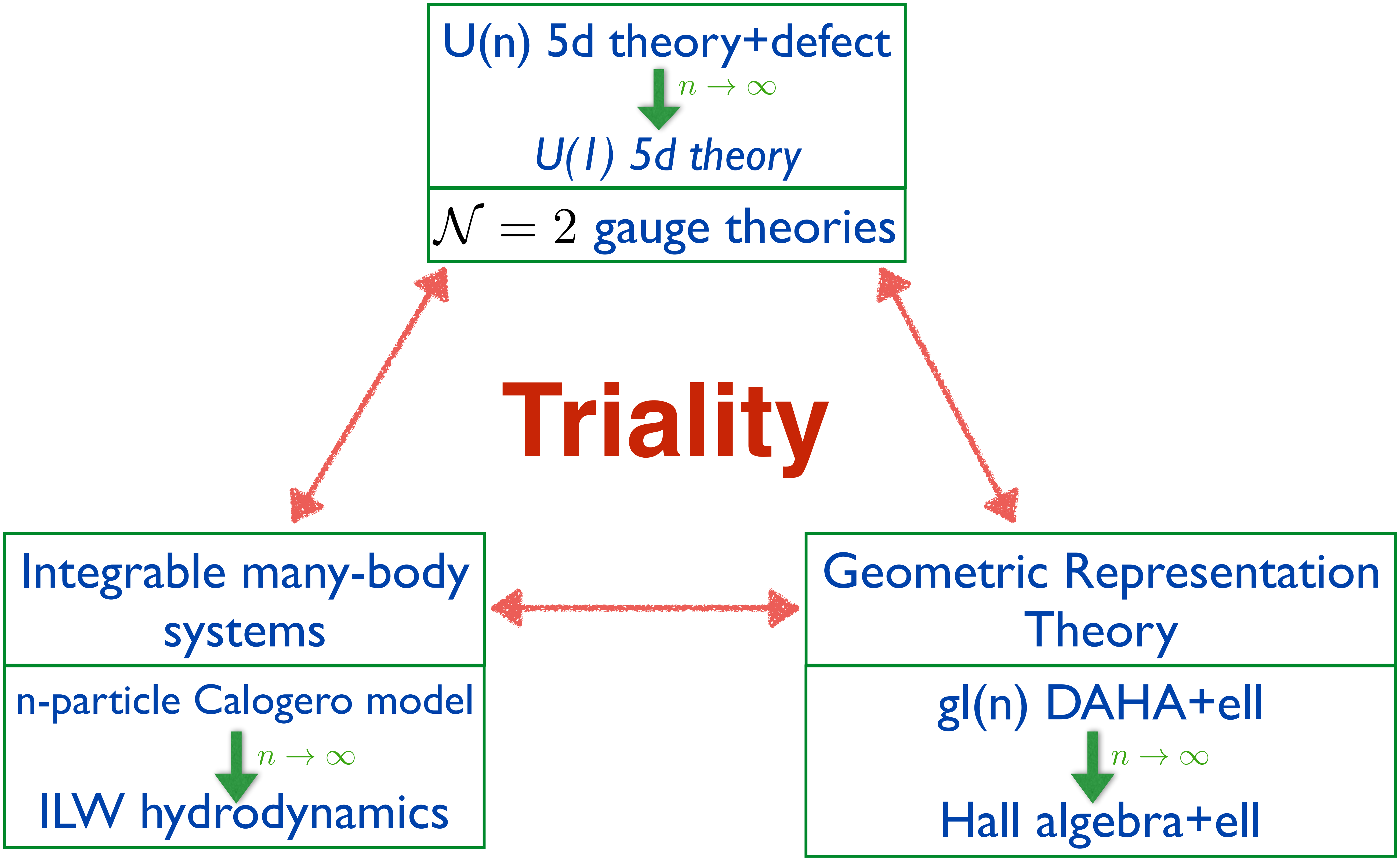

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N=2 Gauge Theories

We focus on N=2 gauge theories which have Seiberg-Witten description in IR

At the moment we have plethora of exact results for those theories thanks to Nekrasov's computation of instanton partition functions

Nekrasov's original work has been greatly extended in to:
- various supergravity backgrounds (e.g. spheres)
- quiver gauge theories
- five and six-dimensional theories on $X_D = \mathbb{R}^4 \times \Sigma$
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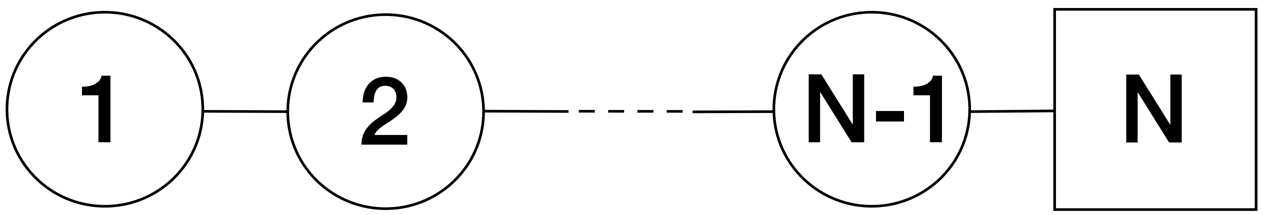
We shall study theories with adjoint matter on

$$X_3 = \mathbb{C}_{\epsilon_1} \times S^1_\gamma \qquad\qquad X_5 = \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times S^1_\gamma$$

3d Theory

$\mathcal{N} = 2^*$ quiver gauge theory on $X_3 = \mathbb{C}_{\epsilon_1} \times S^1_\gamma$ T[U(N)]



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Lagrangian depends on twisted masses μ_i and FI parameters τ_i

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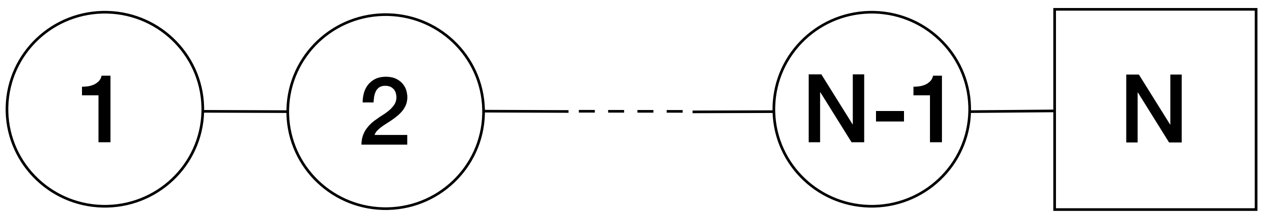

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Partition function computed by localization for N=2

$$\mathcal{B} \sim {}_2\phi_1\left(t, t\frac{\mu_1}{\mu_2}, q\frac{\mu_1}{\mu_2}; q; \frac{\tau_1}{\tau_2}\right) \qquad q = e^{\epsilon_1}$$

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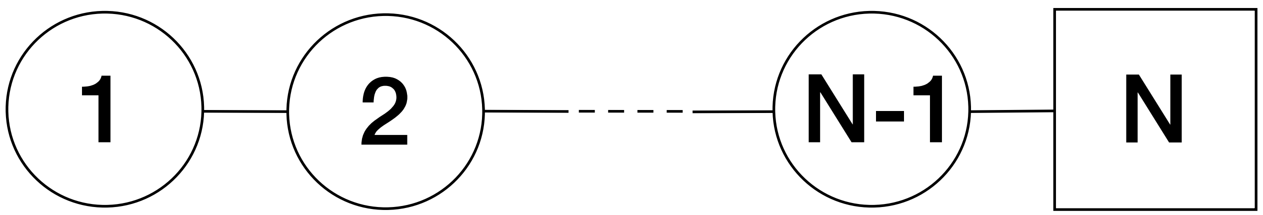

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$q=e^{\epsilon_1}$

is the eigenstate of the trigonometric Ruijsenaars-Schneider system!

$$D^{(1)}\mathcal{B}=(\mu_1+\mu_2)\mathcal{B}$$

$$D^{(1)}\sim\sum_{i\neq j}\frac{t\tau_i-\tau_j}{\tau_i-\tau_j}e^{\hbar\partial_{\log\tau_i}}$$

3d A-type quiver

For T[U(N)] quiver

$$D^{(k)}\mathcal{B} = \left\langle W_k^{U(n)} \right\rangle \mathcal{B}$$

In other words, the eigenvalue of tRS Hamiltonian is a VEV of background Wilson loop around the compact circle

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The Hamiltonians themselves are certain quantizations of the 't-Hooft-vortex loops in the corresponding representation

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We have just constructed a (complex) representation of the double affine Hecke algebra (DAHA)
tRS Hamiltonians form a subalgebra

[PK Gukov in prog]
[Cherednik]
[Oblomkov]

Elliptic Generalization

3d theory describes trigonometric model. How can we generalize the construction to describe the elliptic model?

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Couple 3d theory to 5d theory whose Seiberg-Witten solution gives elliptic Ruijsenaars model

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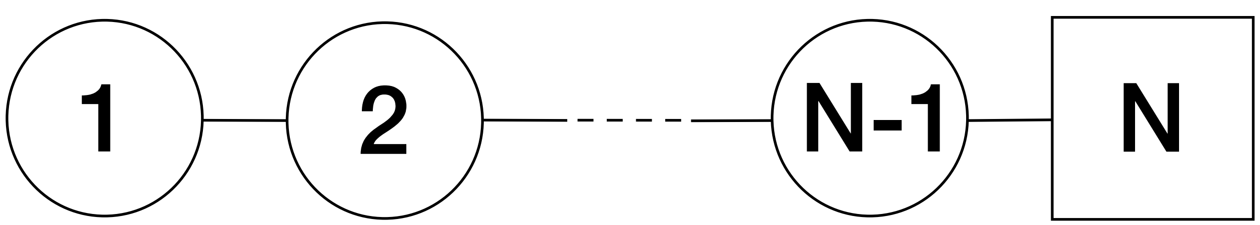
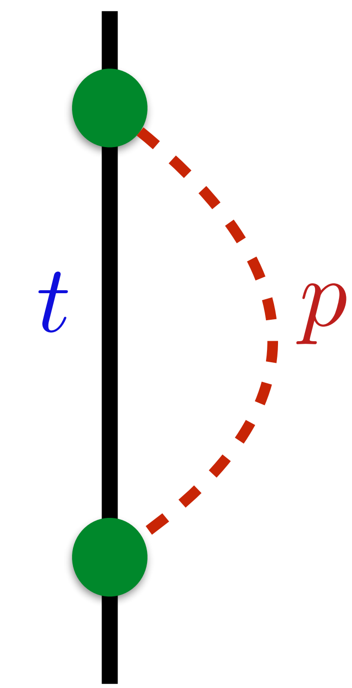
Gauging global symmetry of 3d theory
by gauge group of bulk 5d theory on $X_5 = \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times S^1_\gamma$



5d U(N)
N=1*



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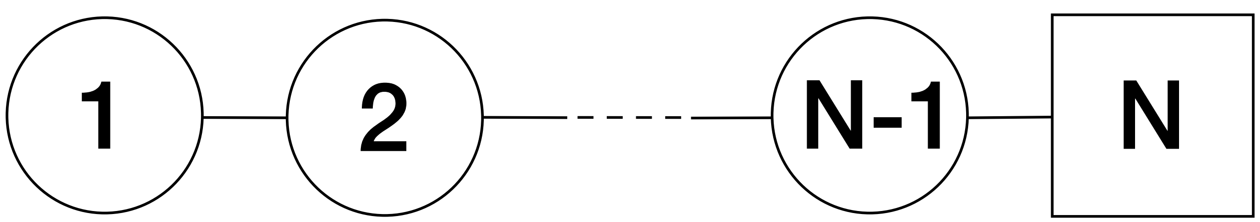

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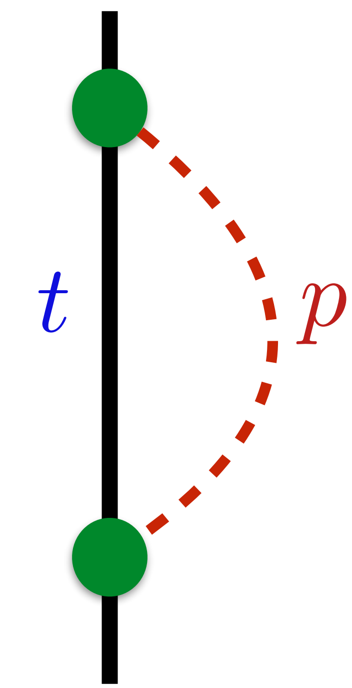

$$D^{(1)}_{p,q,t} \sim \sum \frac{\theta(t\frac{\tau_i}{\tau_j}|p)}{\theta(q\frac{\tau_i}{\tau_j}|p)} e^{\hbar\partial_{\log \tau_i}}$$

$$D^{(k)}_{p,q,t}\mathcal{Z}^{5\mathrm{d}/3\mathrm{d}} \underset{\epsilon_2 \to 0}{=} \left\langle W^{U(n)}_{\Lambda^k} \right\rangle \mathcal{Z}^{5\mathrm{d}/3\mathrm{d}}$$

Gauge/Integrability duality

quantum eRS model	**5d/3d theory**
number of particles n	rank 3d flavor group / 5d gauge group
particle positions τ_j	3d Fayet-Iliopoulos parameters
interaction coupling t	3d $\mathcal{N}=2^*$ / 5d $\mathcal{N}=1^*$ deformation $e^{-i\gamma m}$
shift parameter q	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
elliptic deformation p	5d instanton parameter $Q=e^{-8\pi^2\gamma/g_{YM}^2}$
eigenvalues	$\langle W_{\square}^{U(n)}\rangle$ for 5d $U(n)$ in NS limit
eigenfunctions	$Z_{\text{inst}}^{5d/3d}$ in NS limit at fixed μ_a

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Now we study large-n behavior of the operators (eigenvalues) and the eigenfunctions

Mapping States

Consider partition λ of $k < n$ (assume p=0)

Specify $\mu_a = q^{\lambda_a} t^{n-a} \quad , \quad a = 1, \ldots, n$ for T[U(n)] theory

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Partition function series truncates to Macdonald polynomials!

$$D^{(1)}_{n,\vec{\tau}}(q,t) P_\lambda(\vec{\tau}; q, t) = E^{(\lambda;n)}_{tRS} P_\lambda(\vec{\tau}; q, t)$$

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E.g. k=2

$$\mathcal{B}(\tau_1, \tau_2; t^{-1/2} q, t^{1/2} q) = P_{\square\square}(\tau_1, \tau_2; q, t)$$

$$\mathcal{B}(\tau_1, \tau_2; t^{-1/2}, t^{-1/2} q^2) = P_{\substack{\square \\ \square}}(\tau_1, \tau_2 | q, t) \,.$$

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Their exact form depends on n

$$P_{(2,0)}(\tau_1, \tau_2; q, t) = \tau_1 \tau_2 + \frac{1 - qt}{(1+q)(1-t)}(\tau_1^2 + \tau_2^2)$$

Change of Variables

However, after change of variables $p_m = \sum_{l=1}^{n} \tau_l^m$

Macdonald polynomials depend only on k and the partition

$$P_{\square\square} = \frac{1}{2}(p_1^2 - p_2)\,, \qquad P_{\substack{\square\\\square}} = \frac{1}{2}(p_1^2 - p_2) + \frac{1-qt}{(1+q)(1-t)}p_2$$

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Starting with Fock vacuum $|0\rangle$

Construct Hilbert space $a_{-\lambda}|0\rangle \longleftrightarrow p_\lambda$

for each partition $a_{-\lambda}|0\rangle = a_{-\lambda_1}\cdots a_{-\lambda_l}|0\rangle$

Free boson realization $$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}$$

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Vortex series encodes all states! Now need to describe eigenvalues

Free Boson Realization

Introduce vertex operators [Ding Iohara]

$$\eta(z) =: \exp\left(-\sum_{k\neq 0}\frac{1-t^k}{k}a_k z^{-k}\right):$$

$$\phi(z) = \exp\left(\sum_{n>0}\frac{1-t^n}{1-q^n}a_{-n}\frac{z^n}{n}\right)$$

Define $\phi_n(\tau) = \prod_{i=1}^{n}\phi(\tau_i)$

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Define $\phi_n(\tau) = \prod_{i=1}^{n}\phi(\tau_i)$ then

$$[\eta(z)]_1\phi_n(\tau)|0\rangle = \left[t^{-n} + t^{-n+1}(1-t^{-1})D^{(1)}_{n,\vec{\tau}}(q,t)\right]\phi_n(\tau)|0\rangle$$

Assuming |t|<1

$$\mathcal{E}_1^{(\lambda)} = \lim_{n\to\infty}\left[t^{-n+1}(1-t^{-1})E^{(\lambda;n)}_{tRS}\right]$$

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For elliptic model replace [Feigin Hashizume Hoshino Shiraishi Yanagida]

$$\eta(z;pq^{-1}t) = \exp\left(\sum_{n>0}\frac{1-t^{-n}}{n}\frac{1-(pq^{-1}t)^n}{1-p^n}a_{-n}z^n\right)\exp\left(-\sum_{n>0}\frac{1-t^n}{n}a_n z^{-n}\right)$$

Free Boson Realization

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Free Boson Realization

From gauge theory we can compute

$$\frac{(pt^{-1};p)_\infty (ptq^{-1};p)_\infty}{(p;p)_\infty (pq^{-1};p)_\infty} E^{(\lambda;n)}_{eRS}(p) = \left\langle W^{U(1)}_{\square} \right\rangle E^{(\lambda;n)}_{eRS}(p) = \left\langle W^{U(n)}_{\square} \right\rangle \Big|_{\lambda}$$

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Heisenberg algebra appears in the study of moduli space of U(1) (non-commutative) instantons

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Instanton - KK monopole propagating along the compact circle

KK modes yield different topological sectors

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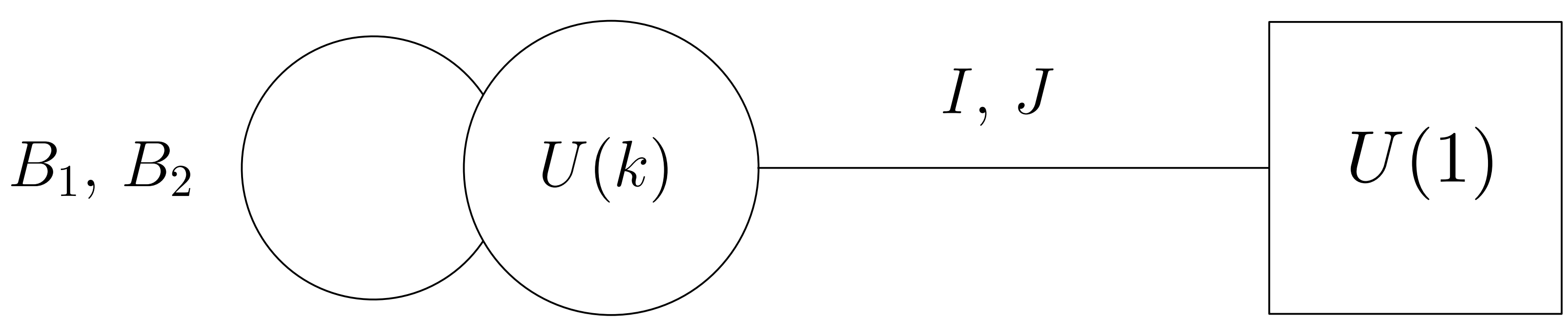
Higgs branch of the 3d N=2 ADHM quiver gauge theory on $\mathbb{C} \times S^1_\gamma$



$\mathcal{M}_{k,1}$

Quantum Cohomology

Using supersymmetry we can effectively describe quantum cohomology (K-theory) of the instanton moduli space $\mathcal{M}_{k,1}$

We need to find the twisted chiral ring of the ADHM gauge theory— Jacobian ring for effective twisted superpotential

$$H_T^\bullet(\mathcal{M}_{k,1}) \simeq \frac{\{\sigma_1, \ldots \sigma_s\}}{\{\partial\widetilde{\mathcal{W}}/\partial\sigma_s = 0\}}$$

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where $\sigma_s = e^{i\gamma\Sigma_s},\ q = e^{i\gamma\epsilon_1},\ t = e^{-i\gamma\epsilon_2}$ $\quad \widetilde{p} = e^{-2\pi\xi}$ FI coupling

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*Calogero **Hamiltonian** contains the operator of **quantum multiplication** in small **quantum cohomology** ring of the instanton moduli space*

The Duality

Eigenvalues at large-n

[PK Sciarappa]

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In other words there exists a stable limit of the equivariant Chern character of the universal bundle over the U(n) instanton moduli space in terms of the same character only for U(1) instantons

The Duality

Eigenvalues at large-n

[PK Sciarappa]

$$\left\langle W_{\square}^{U(n)} \right\rangle \Big|_{\lambda} \sim \mathcal{E}_1^{(\lambda)} = 1 - (1-q)(1-t^{-1}) \sum_s \sigma_s \Big|_{\lambda}$$

Wilson line VEV becomes an equivariant Chern character for $\mathcal{M}_{k,1}$

In other words there exists a stable limit of the equivariant Chern character of the universal bundle over the U(n) instanton moduli space in terms of the same character only for U(1) instantons

elliptic RS	**3d ADHM theory**	**3d/5d coupled theory, $n \to \infty$**
coupling t	twisted mass $e^{-i\gamma\epsilon_2}$	5d $\mathcal{N} = 1^*$ mass deformation $e^{-i\gamma m}$
quantum shift q	twisted mass $e^{i\gamma\epsilon_1}$	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
elliptic parameter p	FI parameter $\widetilde{p} = -p/\sqrt{qt^{-1}}$	5d instanton parameter Q
eigenstates λ	ADHM Coulomb vacua	5d Coulomb branch parameters
eigenvalues	$\langle \mathrm{Tr}\,\sigma \rangle$	$\langle W_{\square}^{U(\infty)} \rangle$ in NS limit $\widetilde{\epsilon}_2 \to 0$

Mathematical Results

[Schiffmann Vasserot]

Hall algebra as large-n limit of DAHA

Trigonometric RS at large n $$\lim_{n\to\infty} K_T(T^*\mathbb{F}_n) \simeq K^{\mathrm{cl}}_{q,t}\left(\widetilde{\mathcal{M}_1}\right)$$

$$\widetilde{\mathcal{M}_1} = \bigoplus_{k=0}^{\infty} \mathcal{M}_{1,k}$$ Instanton moduli space

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No mathematical object is known to describe spectrum of elliptic RS

Our proposal

$$\mathcal{E}^Q_T(T^*\mathbb{F}_n) := \mathbb{C}[p_i^{\pm 1}, \tau_i^{\pm 1}, Q, t, \mu_i^{\pm 1}]/\mathcal{I}_{\mathrm{eRS}}$$

Large-n limit

$$\lim_{n\to\infty} \mathcal{E}^Q_T(T^*\mathbb{F}_n) \simeq K_{q,t}\left(\widetilde{\mathcal{M}_1}\right)$$

Open questions

Relationships between different elliptic deformations
Physics construction for elliptic cohomology

Knot homology

What happens for 6d theories at large n?   Holography?